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N-dimensional Laplace transformations and their applications in partial differential equations

by

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This dissertation is ***dedicated to***:

My ***Mother*** and the memory of my late ***Father***
with ***Gratitude***

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ABSTRACT

This dissertation focuses on the theoretical and computation aspects of N-dimensional Laplace transformation pairs, for $N \geq 2$. Laplace transforms can be defined either as a unilateral or a bilateral integral. We concentrate on the unilateral integrals. We have successfully developed a number of theorems and corollaries in N-dimensional Laplace transformations and inverse Laplace transformations. We have given numerous illustrative examples on applications of these results in N and particularly in two dimensions. We believe that these results will further enhance the use of N-dimensional Laplace transformation and help further development of more theoretical results.

Specifically, we derive several two-dimensional Laplace transforms and inverse Laplace transforms in two-dimension pairs. We believe most of these results are new. However, we have established some of the well-known results for the case of commonly used special functions.

Several initial boundary value problems (IBVPs) characterized by non-homogenous linear partial differential equations (PDEs) are explicitly solved in Chapter 4 by means of results developed in Chapters 2 and 3. In the absence of necessitous three and N-dimensional Laplace transformation tables, we solve these IBVPs by the double Laplace transformations. These include non-homogenous linear PDEs of the first order, non-homogenous second order linear PDEs of Hyperbolic and Parabolic types.

Even though multi-dimensional Laplace transformations have been studies extensively since the early 1920s, or so, still a table of three or N-dimensional Laplace transforms is not available. To fill this gap much work is left to be done. To this end, we have established several new results on N-dimensional Laplace transforms as well as inverse Laplace transforms and many more are still under our investigation. A successful completion of this task will be a significant endeavor, which will be extremely beneficial to the further research in Applied Mathematics, Engineering and Physical Sciences. Especially, by the use of multi-dimensional Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in n independent variables. This algebraic equation can be used to obtain the solution of the original PDE.

CHAPTER 1. INTRODUCTION AND PRELIMINARIES

1.1 Literature Review

The modern theory of the "Heaviside Operational Calculus" is almost universally based on the Laplacian transformation. This basis places the operational method on a rigorous plane and extends the generality of the type of problems which may be solved by operational methods.

The historical development of operational calculus may be summarized briefly by stating that it can be divided into the following four main divisions.

1. *The formal theory of operators.* This phase begins with Gottfried Wilhelm Leibnitz (1646-1716) in which he noticed certain striking analogies between algebraic laws and the behavior of differential and integral operators. The work was further carried on by Joseph Louis Lagrange (1736-1813) and his successors.

2. *The second period. This period was marked by the development of operational processes to the following fields:*

(i) The theory of finite differences.

(ii) The symbolic methods of fractional differentiation.

(iii) The use of symbolic methods in the calculus of finite differences and differential equations.

The leaders during this period were: P.S. Laplace (1749-1827) [62], George Boole (1815-1864), R. Murphy (1806-1843), R. Carmichael (1828-1861). Books embodying the theory were published by G. Boole and R. Carmichael and most of the theorems presented had their modern shape.

3. *The Heaviside period.* This movement of the theory was initiated through the work of Oliver Heaviside (1850-1925) who developed its earlier concepts and applied them successfully to problems dealing with almost every phase of Physics and applied mathematics. These methods which have been proved so useful to engineers are now collected under the name of Heaviside Operational Calculus [50]. In spite of his notable contributions, Heaviside's development of the Operational Calculus was largely empirical and lacking in mathematical rigor. Many electrical engineers hastened to explain certain of Heaviside's rules. Many papers of this explanatory character appeared during the period 1910-1925. Prominent workers of this period were Louis Cohen, E. J. Berg, H. W. March, V. Bush, W. O. Pennell, and J. J. Smith.

4. *The rigorous period.* Operational Calculus again started attracting the notice of mathematicians in the early 1920s Bromwich (1875-1930) was the first to explain, and to a certain extent justify, Heaviside's methods. He made use of the theory of Functions of Complex variable [10].

After Bromwich, J. R. Carson contributed substantially to the theory [16]. He demonstrated that Heaviside's operational method can be fully substantiated, starting from the Laplace transformation, which expresses $f(p)$ in terms of the function $h(t)$, by means of the integral equation

$$f(p) = p\int_0^\infty \exp(-pt)h(t)dt. \tag{1.1.1}$$

In this relation $h(t)$ is called the *original* and $f(p)$ the *image*. Carson's work exercised a considerable influence on subsequent studies in the field of establishing Operational Calculus. The credit for drawing the attention of mathematicians to the Laplace transformation is essentially his.

Van der Pol, in his exposition on operational calculus, also depended on the Laplace integral. Van der Pol and Bremmer [104] considered the case where the lower limit of the integral in (1.1.1.) is replaced by $-\infty$. So that the transform becomes two-sided and admits the form

$$F(p) = p\int_{-\infty}^{\infty} \exp(-tp)h(t)dt. \tag{1.1.2}$$

Before Carson and Van der Pol, however, D. V. Doetsch had been using the same idea, though he multiplied by $\exp(-pt)$ instead of $p\exp(-pt)$. In this language now customary, he applied the Laplace transformation

$$f(p) = \int_{0}^{\infty} \exp(-pt)F(t)dt, \;\; p > 0. \tag{1.1.3}$$

to the differential equations of his problems, including the boundary conditions . He also made an important change in introducing a new symbol in the " subsidiary equations, " as the operational equations are now frequently called.

Subsequent investigations were almost completely found on the Laplace transformation. The theatrical side was developed by, besides Doetsch [45], D. V. Widder in his book, *The Laplace Transform* [112]. Various aspects of Operational Calculus are dealt with in the works of Bromowich and Van der Pol that appeared much later. It is also worth mentioning in this connection the works of Vannover, V. Bush, K. F. Niessen, P. Humbert, M. Harder, H. S. Carslaw [13], L. A. Pipes [75], H.T Davis [35], Ruel V. Churchill [19], N.W. McLachlan [68], K. W. Wagner, Parodi, Colombo, etc.

Various refinements of a basic nature are due to P. Levy and H. Jeffreys.

Operational Calculus made further advances during the forties. While dealing with the spectral theory of linear operators, A. I. Plessner [83] reinforced the foundations of Operational Calculus. V. A. Ditkin [42] extended the results of Plessner.

In the latter half of the forties, the researches of I. Z. Shtokalo [93], [94], [95] and [96] in Operational Calculus were published. These extended Operational Calculus to new classes of linear differential equations with periodic, quasi-periodic and almost-periodic coefficients.

From the forties to the sixties much work has been done in sharpening the Operational Calculus for attacking concrete practical problems. For example, B. V. Bulgakov and I. A. Lurye have investigated the applicability of Operational Calculus to some problems in mechanics. The work of V. I. Krylov and others deals with the digital conversion of the Laplace transform. The investigations of M. I. Kantorowicz are devoted to the applicability of operational methods in the handling of non-stationary phenomena in electrical circuits.

The exploitation of the resources of Operational Calculus in the field of electrical technology is due to M. J. Yuriev, K. V. Krug and E. A. Mirovich; in radio technology to C. I. Evtyanov; in heat technology to A. V. Likov; and in mathematical physics to A. J. Povzner.

The theory of automatic control repellents another avenue for the application of Operational Calculus. Important results in this field have already been realized by A. A. Andronov and his co-workers. As remarked by Andronov, Operational Calculus constitutes the very alphabet of modern automatic and remote control technologies. The applicability of Operational Calculus in the theory of automatic control has expanded in various directions, especially in resolving the problems of combustion mechanics. So, O. M. Krizhanovskii has examined questions related to the use of Operational Calculus in the analysis of the functioning of an automatic regulator of a mine-lift. O. A. Zalesov has used Operational Calculus in the theory of an overturned mine-cage.

Important results in the theory and application of operational methods are due to K. G. Valiv and his co-workers. Some of their research has been extended by the techniques developed by I. Z. Shtokalo.

Essentially four methods have been used in discussing the Heaviside Calculus. They are:

(i) Direct use of formal operators.

(ii) Complex line integrals.

(iii)The Laplacian transformation.

(iv)The Fourier integral.

The method of Laplacian transformation appears to be the most general and natural.

In spite of the numerous applications of one-dimensional Laplace transformation, the idea naturally arose of generalizing the transform functions of two variables. According to Ditkin and Prudnikov [43] during the 1930s short notes by P. Humbert [54], [55] and by P. Humbert and N. W. McLachlan [56] on the Operational Calculus in two variables based on the two-dimensional Laplace transformation appeared. However, T. A. Estrin and T. J. Higgins [48] pointed out that, “double Laplace transforms” were introduced by Van der Pol and K. F. Niessen [105]. They used by P. Humbert [55] in his study of hyper geometric functions; by J. C. Jaeger [58] to solve boundary value problems in heat conduction; by N. A. Shastri [92]. In the works of P. Delerue [38], [39] and Voelker and Doetsch [107] the methods of the Operational Calculus in several variables were successfully applied to the solution of differential equations by the study of the properties of special functions. Dorothy L. Bernstein [7] wrote her Ph.D. dissertation on The Theory of Double Laplace Integral. A few years later G. A. Coon and D. L. Bernstein [18] published a paper on double Laplace transformation. Also there are contributions by A. Duranona Y. Vedia and C. A. Trejo [47].

D. A. George explored the use of the Voltera [106] series in the building block approach to control systems and demonstrated the usefulness of the multi-dimensional Laplace transform. J. K. Lubbock and V. S. Bansal [64] applied multi-dimensional Laplace transforms for the solution of non-linear equations.

In 1962, V. A. Ditkin and A. P. Prudnikov [43] discussed the fundamental properties of two-dimensional Laplace transformation as the basis of Operational Calculus in two Variables. Also the two-dimensional Laplace transformation of functions and its applications were considered in the books of H. Delavault [37] and J. Hladik [52]. The n-dimensional case is treated in the booklet of L. G. Smyshlyaeva [98] and the most recent book written by Y. A. Brychkov et al. [11]. More recently, a number of results on two-, three-, and n-dimensional Laplace transforms were proposed by R. S. Dahiya [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [32] and [33], J. Saberi-Nadjafi [89], J. Saberi-Nadjafi and R. S. Dahiya [90] and doubtless by others.

1.2. Explanation of Dissertation Format

This introductory chapter deals with a brief review of literature on one-, two- and N-dimensional Laplace transformations. Next, we state the objectives of the present research work and explain the notations and make a list of all special functions used in this work. In last two sections of this chapter, we recapitulate the definitions, theorems in one-, two- and N-dimensional Laplace transformations, and rules of operational calculus .

In Chapter 2, a number of new and useful theorems and corollaries on N-dimensional Laplace transformation and N-dimensional inverse Laplace transformation were developed. These theorems are proved according to an idea obtained from papers of R.S. Dahiya [21] and [30]. Several examples are given to show these results can be used to obtain new two and N-dimensional Laplace transformation pairs.

In Chapter 3, we established several new formulas for calculating Laplace transformation pairs of N-dimensions from a known one-dimensional Laplace transform. We have given several examples on applications of these results in N and two-dimensions.

Several initial boundary value problems (IBVPs) characterized by non-homogenous linear partial differential equations (PDEs) are explicitly solved in Chapter 4 by means of results developed in Chapters 2 and 3. In the absence of necessitous three and N-dimensional Laplace transformation tables, we focused on two dimensions and solved these IBVPs by the double Laplace transformations. These include non-homogenous linear PDEs of the first order, non-homogenous second order linear PDEs of Hyperbolic type as well as Parabolic type.

1.3. Motivation and Objectives of the Dissertation

The Laplace transform, it can be fairly said, stands first in importance among all integral transforms; for which there are many specific examples in which other transforms prove more expedient. The Laplace transform is the most powerful in dealing with both initial-boundary-value problems (IBVPs) and transforms.

Before we proceed to a detailed exposition of this section, it is of some interest to list some of the better known uses of the Laplace transformation theory in applied mathematics.

1. *The solution of ordinary differential equations with constant coefficient:*

The Laplace transform method is particularly well adapted to the solution of differential equations whose boundary conditions are specified at one point. The solution of differential equations involving functions of an impulsive type may be solved by the use of Laplace transformation in a very efficient manner. Typical fields of application are the following:

(a) Transient and steady-state analysis of electrical circuits. [15], [20], [59], [68],[69], [106].

(b) Applications to dynamical problems (impact, mechanical vibrations, acoustics, etc.) [61], [68] , [69], [102], [20].

(c) Applications to structural problems (deflection of beams, columns, determination of Green's functions and influence functions. [14], [68], [69], [102].

2. *The solution of linear differential equations with variable coefficients* [20], [106].

3. *The solution of linear partial differential equations with constant coefficients*:

One of the most important uses of the Laplace transformation theory is its use in the solution of linear partial differential equations with constant coefficients that have two or more independent variables. Typical physical problems that may be solved by the procedure of this method are the following:

(a) Transient and steady-state analysis of heat conduction in solids [14], [59], [20].

(b) Vibrations of continuous mechanical systems [14], [68], [20].

(c) Hydrodynamics and fluid flow [14], [68], [69], [91].

(d) Transient analysis of electrical transmission lines and also of cables [15], [14], [106], [108], [68].

(e) Transient analysis of electrodynamics fields [61], [68].

(f) Transient analysis of acoustical systems [68].

(g) analysis of static defalcation of continuous systems (strings, beams, plates) [14], [20].

4. *The solution of linear difference and difference - differential equations:*

The Laplace transformation theory is very useful in effecting the solution of linear difference or mixed linear-difference-differential equations with constant coefficients [14], [68], [77], [106].

5. *The solution of integral equations of the convolution, or Faltung,type:* [15], [45], [20].

6. *Application of the Laplace transform to the theory of prime numbers.*

7. *Evaluation of definite integrals*, [106].

8. *Derivation of asymptotic series* [15], [58].

9. *Derivation of power series* [15], [58].

10. *Derivation of Fourier series* [58], [68].

11. *Summation of power series* [58].

12. *The summation of Fourier series* [78].

13. *The solution of non- linear ODEs* [74], [76], [79], [80], [81].

14. *The use of multi-dimensional Laplace transformations to solve linear PDEs with constant coefficients::*

The usual operational method of solving boundary value problems in time and space variables transforms the PDEs and its boundary conditions with respect to time, the space variables being held constant; solves the resulting ordinary or partial differential equation by classical means, the transform parameter being treated as a constant; recognizes the resulting expression in the transform parameter as the single Laplace transform of the desired solution; and effects the required inversion. Apparently, the use of a single Laplace transformation in this manner is not as advantageous as it is in the solution of ODEs, because an ordinary or partial differential equation yet remains to be solved after the single transformation.

By the use of multiple Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in n independent complex variables. This algebraic equation can be solved for multiple transform of the solution of the original PDE. Multiple inversion of this transform then gives the desired solution[32], [33], [48], [58], [106], [107].

The analytical difficulty of evaluating multiple inverse transforms (Formula 1.6.2.1 on page 32) increases with the number of independent variables, to the end that a fairly comprehensive knowledge of contour integration may be needed to reach the desired solution. The difficulty in obtaining inverse Laplace transforms using the techniques of complex analysis lead to continued efforts in expanding the transform tables and in designing algorithms for generating new inverses from the known results, using some other techniques.

The primary objective of this dissertation is to establish several new results for calculating Laplace transformation and inverse Laplace transformation pairs of N-dimensions from one-dimensional Laplace transformations. Next we applied these results to a number of commonly used special functions to obtain a new Laplace transformation in two and N-dimensions. Specifically, we derived some of the well-known two-dimensional Laplace transformation pairs, using some of our established results in Chapters 2 and 3.

In Chapter 4, several IBVPs characterized by non-homogenous linear PDEs are explicitly solved by means of using some of the developed results in previous chapters. These IBVPs include non-homogenous linear PDEs of the first order, non-homogenous second order linear PDEs of Hyperbolic as well as Parabolic types.

Chapter 5, summarized the major contributions of this dissertation, and traces possible direction for future research in this area. A list of references is included at the end.

1.4. Notations and Special Functions

1.4.1. Notations

We begin this section with the description of notations, that we will use in this dissertation:

For any real n-dimensional variable $\bar{x} = (x_1, x_2, \ldots, x_n)$,and for any complex n-dimensional variable $\bar{s} = (s_1, s_2, \ldots, s_n)$, we denote $\overline{x^\nu} = (x_1^\nu, x_2^\nu, \ldots, x_n^\nu)$ and $\overline{s^\nu} = (s_1^\nu, s_2^\nu, \ldots, s_n^\nu)$ where ν is any real number.

Let $p_k(\bar{x})$ or $p_k(\bar{s})$ be the kth symmetric polynomial in the components x_k or s_k of x or s respectively. Then for $\bar{x} = (x_1, x_2, \ldots, x_n)$and $\bar{s} = (s_1, s_2, \ldots, s_n)$, we denote

(a1) $p_1(\overline{x^\nu}) = x_1^\nu + x_2^\nu + \cdots + x_n^\nu = \sum_{j=1}^{n} x_j^\nu$

(a2) $p_1(\overline{s^\nu}) = s_1^\nu + s_2^\nu + \cdots + s_n^\nu = \sum_{j=1}^{n} s_j^\nu$

(a3) $p_1(\overline{x_{\substack{ij \\ i<j}}{}^\nu}) = x_1{}^\nu + x_2{}^\nu + \ldots + x_{i-1}{}^\nu + x_{i+1}{}^\nu + \ldots + x_{j-1}{}^\nu + x_{j+1}{}^\nu + \ldots + x_n{}^\nu$

(b1) $p_n(\overline{x^\nu}) = x_1^\nu \cdot x_2^\nu \cdots x_n^\nu = \prod_{j=i}^{n} x_j^\nu$

(b2) $p_n(\overline{s^\nu}) = s_1^\nu \cdot s_2^\nu \cdots s_n^\nu = \prod_{j=i}^{n} s_j^\nu$

Also we shall write

(i) $\bar{x}\cdot\bar{s}=\sum_{j=1}^{n}x_j s_j$

(ii) $\Re e\, s$ =Real part of a complex number s

(iii) $\Re e\,[p_1(\overline{s^v})]$= Real part of a complex number $(s_1^v+s_2^v+\dots+s_n^v)$

As usual we denote by $\mathbf{N}$ the set of natural numbers, $\mathbf{N}=\{1,2,\ldots\}$ and $\mathbf{N}_0=\mathbf{N}\cup\{0\}$. By $\mathbf{R}^n$ we denote the *n-dimensional Euclidean space,* $n\in\mathbf{N}$. Analogously, we denote *unitary space* by $\mathbf{C}^n$. By subsets $\mathbf{R}_+^n$and $\overline{\mathbf{R}}_+^n$ of $\mathbf{R}^n$ we mean

$$\mathbf{R}_+^n=\{\bar{x}:\bar{x}\in\mathbf{R}^n,\ \bar{x}>0\}$$
$$\overline{\mathbf{R}}_+^n=\{\bar{x}:\bar{x}\in\mathbf{R}^n,\ \bar{x}\geq 0\}.$$

Let ω be an open subsets of $\mathbf{R}^n$. The linear space of all measurable functions f defined on ω for which the expressions

$$\int_\omega |f(\bar{x})|^p\,dx_1dx_2\ldots dx_n=\int_\omega |f(\bar{x})|^p\,d\bar{x},\ \ p\in\mathbf{R},\quad p\geq 1,$$

are finite is denoted by $L_p(\omega)$, which is equipped with the norms

$$\|f\|_{L_p(\omega)}=\left[\int_\omega |f(\bar{x})|^p\,dx_1dx_2\ldots dx_n\right]^{\frac{1}{p}}=\left[\int_\omega |f(\bar{x})|^p\,d\bar{x}\right]^{\frac{1}{p}},\ \ 1\leq p\leq\infty,$$

is a complete normed space, i.e., *Banach space.*

By $L_1[\ \mathbf{R}_+^n;\exp(-\bar{a}.\bar{x})]$ we mean the linear space of all measurable function f defined on $\mathbf{R}_+^n$ for which the following conditions hold:

(i) $\int_\omega |f(\bar{x})|d\bar{x}$ are finite

$$|f(\bar{x})| \leq M \exp(\bar{a}.\bar{x}),$$

for all measurable functions f on $\mathbf{R}^n_+$ where M and $\bar{a}$ are positive constants.

Next, let us recall the space $L_1^{Loc}(\omega)$. The elements $f \in L_1^{Loc}(\omega)$ are called locally integrable on ω, and it consists of all functions f such that $f \in L_1(\omega')$ for every ω' such that $\omega' \subset \omega$.

If $u(x,y)$ be a function of two variables x and y, we denote

$$\frac{\partial u(x,y)}{\partial x} = u_x, \quad \frac{\partial u(x,y)}{\partial y} = u_y,$$
$$\frac{\partial^2 (x,y)}{\partial x \partial y} = u_{xy}, \quad \frac{\partial^2 u(x,y)}{\partial x^2} = u_{xx}, \quad \frac{\partial^2 u(x,y)}{\partial y^2} = u_{yy}.$$

1.4.2. Special Functions

Most of the two or N-dimensional results we have obtained are involving certain special functions. Following is a list of all the special functions used in this dissertation:

Fresnel integrals:

$$C(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^x \frac{\cos u}{u^{\frac{1}{2}}} du$$
$$S(x) = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^x \frac{\sin u}{u^{\frac{1}{2}}} du$$

Cosine integrals:

$$Ci(x) = -\int_x^\infty \frac{\cos u}{u} du$$
$$ci(x) = \int_x^\infty \frac{\cos u}{u} du$$

Parabolic cylinder functions:

$$D_n(x) = (-1)^n \exp(\tfrac{x^2}{4}) \frac{d^n}{dx^n} [\exp(-\tfrac{x^2}{2})] \ , \ n = 0,1,2,\ldots$$

$$D_\nu(x) = \frac{2^{\frac{\nu}{2}+\frac{1}{4}}}{x^{\frac{1}{2}}} W_{\frac{\nu}{2}+\frac{1}{4},\frac{1}{4}}(\frac{x^2}{2})$$

Exponential integrals:

$$Ei(-x) = -\int_x^\infty \exp(-u) \frac{du}{u} \qquad -\pi < \arg x < \pi$$

$$\overline{Ei}(x) = \tfrac{1}{2}[Ei(x+i0) + Ei(x-i0)] \qquad x > 0$$

Error function:

$$Erf(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_0^x \exp(-u^2) du$$

Complementary Error function:

$$Erfc(x) = \frac{2}{\pi^{\frac{1}{2}}} \int_x^\infty \exp(-u^2) du$$

Error function of an imaginary argument:

$$Erfi(x) = \frac{2}{x^{\frac{1}{2}}} \int_0^x \exp(u^2) du$$

(n+1)th derivative of Error function:

$$\frac{d^{n+1}}{dx^{n+1}} Erf(x) = (-1)^n \frac{2}{\pi^{\frac{1}{2}}} \exp(-x^2) H_n(x)$$

$$\frac{d^{n+1}}{dx^{n+1}}Erf(x) = (-1)^n \frac{2}{\pi^{\frac{1}{2}}}\exp(-x^2)H_n(x)$$

Generalized Hypergeometric function:

$${}_pF_q\left[\begin{matrix}(a)_p; \\ (b)_q;\end{matrix} x\right] = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!},$$

where $(a_j)_m = a_j(a_j+1)(a_j+2)\ldots(a_j+m-1),\ j = 1,2,\ldots,p,$
$(a_j)_0 = 1.$

nth derivative of a Hypergeometric function:

$$\frac{d^n}{dx^n}\ {}_pF_q\left[\begin{matrix}a_p; \\ a_q;\end{matrix} x\right] = \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n}\ {}_pF_q\left[\begin{matrix}(a+n)_p; \\ (b+n)_q;\end{matrix} x\right]$$

G-function (Meijer's G function):

$$G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\ldots,a_p \\ b_1,\ldots,b_q\end{matrix}\right.\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s ds$$

Here an empty product is interpreted as unity, $0 \le m \le q,\ 0 \le n \le p,$ and the parameters $a_h,\ b_h$ are such that no pole of $\Gamma(b_j - s),\ j = 1,2,\ldots,m,$ coincides with any pole of $\Gamma(1-a_k+s),\ k = 1,2\ldots,n.$ Thus $(a_k - b_j)$ is not a positive integer. Also $z \neq 0.$

There are three different paths L of integration. For more details see [65 ; pages 143-152].

Hermit Polynomial:

$$H_n(x) = (-1)^n \exp(x^2)\frac{d^n}{dx^n}[\exp(-x^2)],\ n = 0,1,2,\ldots$$

Struve function:

$$H_\nu(x)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{x}{2})^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}$$

Modified Bessel function:

$$I_\nu(x)=\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

Bessel function:

$$J_\nu(x)=\sum_{n=o}^{\infty}\frac{(-1)^n(\frac{x}{2})^{\nu+2n}}{n!\Gamma(\nu+n+1)}$$

Modified Hankel function:

$$K_\nu(x)=\frac{\pi}{2}\frac{I_{-\nu}(x)-I_\nu(x)}{\sin(\nu x)}$$

Whittaker function:

$$M_{\mu,\nu}(x)=x^{\frac{1}{2}+\nu}\exp(-\tfrac{x}{2})\,{}_1F_1\left[\begin{matrix}\frac{1}{2}+\nu-\mu; \\ 2\nu+1;\end{matrix}x\right]$$

Legender polynomial:

$$P_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n,\ n=0,1,2,...$$

Lommel functions:

$$S_{\mu,\nu}(x)=s_{\mu,\nu}(x)+2^{\mu-1}\Gamma\left(\tfrac{\mu-\nu+1}{2}\right)\Gamma\left(\tfrac{\mu+\nu+1}{2}\right)\left[\sin\left(\tfrac{\mu-\nu}{2}\pi\right)J_\nu(x)-\cos\left(\tfrac{\mu-\nu}{2}\pi\right)Y_\nu(x)\right]$$

$$s_{\mu,\nu}(x)=\frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left[\begin{matrix}1 & ; \\ \frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2} & ;\end{matrix}-\frac{x^2}{4}\right]$$

Sine integrals:

$$Si(x) = \int_0^x \frac{\sin u}{u} du$$
$$si(x) = -\int_x^\infty \frac{\sin u}{u} du$$

Heaviside's Unit function:

$$u(x-a) = \begin{cases} 0 & \text{if } x < a \\ 1 & \text{if } x > a \end{cases}$$

Gamma function:

$$\Gamma(x) = \int_0^\infty \exp(-u)\, u^{x-1} du$$

Note: We shall use the following theorem to simplify some of our results in this work.

Duplication Theorem

Let z be a complex number with $z \neq 0, -1, -2, \ldots$, then

$$\Gamma(2z) = \pi^{-\frac{1}{2}} 2^{2z-1} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right).$$

Also, we shall use the following , wherever it would be necessary

$$\Gamma(z+1) = z\Gamma(z),\ z \neq 0, -1, -2, \ldots$$
$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},\ z \in \mathbf{C}$$
$$\Gamma\left(\tfrac{1}{2}\right) = \pi^{\frac{1}{2}},\ \Gamma\left(\tfrac{3}{2}\right) = \tfrac{\pi}{2}^{\frac{1}{2}},\ \Gamma\left(-\tfrac{1}{2}\right) = -2\pi^{\frac{1}{2}},\ \Gamma\left(-\tfrac{3}{2}\right) = \tfrac{4\pi^{\frac{1}{2}}}{3}.$$

Incomplete gamma function:

$$\Gamma(a,x) = \int_x^\infty \exp(-u)\, u^{a-1} du$$

Incomplete gamma function:

$$\gamma(a,x) = \int_0^x \exp(-u)\, u^{a-1} du$$

Note: We shall use the following formulas, wherever it would be necessary.

$$\Gamma(1,x)=\exp(-x),$$
$$\gamma(1,x)=1-\exp(-x),$$
$$\Gamma(\tfrac{1}{2},x^2)=\pi^{\frac{1}{2}}Erfc(x),$$
$$\gamma(\tfrac{1}{2},x^2)=\pi^{\frac{1}{2}}Erf(x),$$
$$\frac{\partial}{\partial \mathrm{x}}\Gamma(a,x)=-x^{a-1}\exp(-x),$$
$$\frac{\partial}{\partial x}\gamma(a,x)=x^{a-1}\exp(-x),$$
$$\Gamma(a+1,x)=a\Gamma(a,x)+x^a\exp(-x),$$
$$\gamma(a+1,x)=a\gamma(a,x)-x^a\exp(-x).$$

Logarithmic derivative of the gamma function:

$$\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}.$$

We denote

$$u(x,0)=\alpha(x),\quad u(0,y)=\beta(y),\quad u(0,0)=c \text{ where } c \text{ is a costant}$$
$$u_y(x,0)=\theta(x),\quad u_x(0,y)=\delta(y)$$

and assume $\alpha(x)$, $\beta(y)$, $\theta(x)$ and $\delta(y)$ are Laplace transformable.

1.5. Recapitulations on One and Two-dimensional Laplace Transformations

1.5.1. One-dimensional Laplace Transformation

Definition of the Laplace transformation:

We define the Laplace transform of $f(x)$ to be the function $F(s)$ given by

$$F(s) = \int_o^\infty \exp(-sx) f(x) dx \qquad (1.5.1.1)$$

for all complex s for which the integral converges.

The integral on the right of (1.5.1.1) is called a Laplace integral. It is improper at the upper limit and may be improper at the lower limit. It is therefore to be understood as

$$\lim_{\substack{w \to \infty \\ d \to o}} \int_d^w \exp(-sx) f(x) dx,$$

where $\omega \to \infty$ and $d \to o$ independent from each other.

We denote the right hand side of (1.5.1.1) by $\mathcal{L}\{f(x);s\}$.Thus (1.5.1.1) may be written in the form

$$\mathcal{L}\{f(x);s\} = F(s) = \int_0^\infty \exp(-sx) f(x) dx \qquad (1.5.1.1')$$

Definition 1.1: The function $f(x)$ is said to be sectionally continuous over the closed interval $a \le x \le b$ if that interval can be divided into a finite number of subintervals $c \le x \le d$ such that in each subinterval

(*i*) $f(x)$ is continuous in the open interval $c < x < d$

(*ii*) $f(x)$ approaches a limit as x approaches each end-points from within the interval; that is, $\lim_{x \to c+} f(x)$ and $\lim_{x \to c-} f(x)$ exist.

Note: We shall always use the term sectionally continuous in the range $x \ge 0$ only. A sectionally continuous function may have infinitely many finite discontinuities in an infinite interval, but it can have only a finite number in any finite interval $0 \le x \le b$ for all $b \ge 0$.

Definition 1.2: The function $f(x)$ is said to be of exponential order as $x \to \infty$ if there exist constants M and a and a fixed $x-value\, x_0$ such that

$$|f(x)| < M \exp(ax), \text{ for } x \ge x_o. \qquad (1.5.1.2)$$

If a is to be emphasized, we say that $f(x)$ is of the order of $\exp(ax)$ as $x \to \infty$. We also write

$$f(x) = O(\exp(ax)),\ x \to \infty,$$

to mean that $f(x)$ is of exponential order, the exponential being $\exp(ax)$, as $x \to \infty$.

A consequence of (1.5.1.2) is $\lim_{x\to\infty} |\exp(a'x) f(x)| = 0,\ (a' > a)$. In particular, when $f(x)$ is of exponential order $\exp(ax)$ then it is also of exponential order $\exp(a'x)$, $(a' > a)$.

The familiar functions $\sin kx$, x^n, and $\exp(kx)$ are examples of functions of exponential order; but the function $\exp(x^2)$ is not of exponential order.

1.5.1.1. The Original Space Ω

The Laplace transformation is a mapping of a set of functions $f, g, \ldots$ defined on the $[0, \infty)$ onto a set of functions $F, G, \ldots$ of a complex variable. The domain of definition of the Laplace transformation is called the *original space* and is denoted by Ω. The range $L\Omega$ of the transformation is called the *image space*. The members of Ω are called original functions, and those of $L\Omega$ image functions.

In the present work, the original space Ω(or class Ω) is taken to consist of all complex valued functions f that satisfy the conditions that are given in definitions (1.1) and (1.2).

Remark 1: The fact that a function $f(x)$ is sectionally continuous or of exponential order is not alone suffocate to insure that it has a Laplace transformation. for example $\exp x^2$ is a continuous function but not of exponential order, does not have a Laplace transformation and $f(x)=x^{-2}$ is of exponential order but the integral fails to exists because of the behavior of the functioning a neighborhood of $x=0$.

These two examples show that neither sectional continuity nor exponential order alone is sufficient to insure that $f(x)$ has a Laplace transform. However, both conditions taken together do suffice.

Theorem 1.1. If $f(x)$ is a function of class Ω, then (1.5.1.1') exists for $\Re e\, s>a$. In fact, $L\{|f(x)|;s\}=\int_0^\infty \exp(-sx)|f(x)|dx$ exists; that is, $\int_0^\infty \exp(-sx)f(x)dx$ is absolutely convergent, for $\Re e\, s>a$.

Remark 2: The above conditions for the existence of the transform are sufficient rather than necessary conditions. The function f may have an infinite discontinuity at $x=0$ for instance, that is $|f(x)|\to\infty$ as $x\to 0$, provided that positive numbers m, N and T exist where $m<1$, such that $|f(x)|<\frac{N}{x^m}$ when $0<x<T$. Then if f otherwise satisfies the above conditions, its transform still exists, because of the existence of the integral

For example, when $f(x)=x^{-\frac{1}{2}}$, its transform can be written, after the substitution of u for $(sx)^{\frac{1}{2}}$, in the form

$$\int_0^\infty x^{-\frac{1}{2}}\exp(-sx)dx=\frac{2}{s^{\frac{1}{2}}}\int_0^\infty \exp(-u^2)du,\ \Re e\ s>0.$$

The last integral has value $\frac{\pi^{\frac{1}{2}}}{2}$; hence

$$L\left\{x^{-\frac{1}{2}};\ s\right\}=\left(\frac{\pi}{s}\right)^{\frac{1}{2}},\ \Re e\ s>0.$$

Theorem 1.2. If $f(x)$ is a function of class Ω, and $L\{f(x);s\}=F(s)$, then

$$\lim_{s\to\infty} F(s)=0.$$

From Theorem 1.2, we conclude that, polynomials in s, $\sin s$, $\cos s$, $\exp(s)$ and $\log s$ can not be Laplace transforms. On the other hand, a rational function is a Laplace transform if the degree of the numerator is less then that of the denominator.

Theorem 1.3. (Fundamental Theorem). If the Laplace integral

$$\int_0^\infty \exp(-sx)f(x)dx\ ,$$

converges for $s=s_0$, then it converges in the open half-plane $\Re e\ s>\Re e\ s_0$, where it can be expressed by the absolutely converging integral

$$(s-s_0)\int_0^\infty \exp[-(s-s_0)]\varphi(x)dx$$

$$(s-s_0)\int_0^{\infty}\exp[-(s-s_0)]\varphi(x)dx$$

with

$$\varphi(x)=\int_0^{x}\exp(-s_0\tau)f(\tau)d\tau.$$

Remark 3: The same conclusion is valid for a Laplace integral which does not converge at s_0 (that is, the limit of $\varphi(x)$ as $x\to\infty$ does not exist), provided $\varphi(x)$ is bounded: $|\varphi(x)|\le M$, for $x\ge 0$.

1.5.2. Two-dimensional Laplace and Laplace-Carson Transformations

Let $f(x,y)$ be a real or complex valued function of two variables, defined on the region $R=\{(x,y):0\le x<\infty, 0\le y<\infty\}$ and integrable in the sense of Lebsgue over an arbitrary finite rectangle $R_{a,b}=\{(x,y):0\le x\le a, 0\le y\le b\}$.

We shall consider the expression

$$F(s_1,s_2;a,b)=\int_0^{a}\int_0^{b}\exp(-s_1x-s_2y)f(x,y)dxdy \qquad (1.5.2.1)$$

where $s_1=\sigma+i\mu$ and $s_2=\tau+iv$ are complex parameters determining a point (s_1,s_2) in the plane of two complex dimensions. Let Ω_2 be the class of all functions $f(x,y)$, such that the following conditions are satisfied for at least one point (s_1,s_2):

(i) The integral (1.5.2.1) is bounded at the point (s_1,s_2) with respect to the variables a and b; i.e.,

$$|F(s_1,s_2;a,b)|<M(s_1,s_2)$$

for all $a\ge 0$ and $b\ge 0$, where $M(s_1,s_2)$ is a positive constant independent of a and b.

(ii) At the point (s_1, s_2)

$$\lim_{\substack{a \to \infty \\ b \to \infty}} F(s_1, s_2; a, b) = F(s_1, s_2)$$

exists. We denote this limit by

$$F(s_1, s_2) = L_2\{f(x, y); s_1, s_2\} = \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 y) f(x, y)\, dx dy \qquad (1.5.2.2)$$

Definition 1.3: The integral (1.5.2.1) is called the two-dimensional Laplace transformation of the function $f(x, y)$.

Remark 1: If the conditions (i) and (ii) are satisfied simultaneously, we will say that the integral (1.5.2.2) converges boundedly in at least one point (s_1, s_2). Thus the class Ω_2 consists of functions for which the integral (1.5.2.2) converges boundedly for at least one point (s_1, s_2). When the integral (1.5.2.2) converges boundedly, we will call $f(x, y)$ the *original function* and $F(s_1, s_2)$ the *image function.*

Remark 2: If the function $f(x, y)$ satisfies the condition

$$|f(x, y)| \le M \exp(hx + ky),$$

for all $x \ge 0, y \ge 0$ (where M, h, k are positive constants), then it is not difficult to verify that $f(x, y)$ belongs to the class Ω_2 at all points (s_1, s_2) for which $\Re e\, s_1 > h$ and $\Re e\, s_2 > k$.

Remark 3: If the function $f(x, y) = f_1(x) f_2(y)$ and the integrals

$$F_1(s_1) = \int_0^\infty \exp(-s_1 x) f_1(x)\, dx, \quad F_2(s_2) = \int_0^\infty \exp(-s_2 y) f_2(y)\, dy$$

exist, then $f(x,y)$ belongs to the class Ω_2 and $F(s_1,s_2) = F_1(s_1)F_2(s_2)$.

Theorem 1.4. If the integral (1.5.2.2) converges boundedly at the point (s_{10}, s_{20}), then it converges boundedly at all points (s_1,s_2) for which $\Re e\,(s_1 - s_{10}) > 0$, $\Re e\,(s_2 - s_{20}) > 0$.

Definition 1.4: We shall denote by D the set of all points (s_1,s_2), for which the integral (1.5.2.2) is boundedly convergent, we call this the region of convergence of the Laplace integral. We remark here that the convergence or divergence of the integral (1.5.2.2) for all real values (s_1,s_2) implies the convergence or divergence for all complex values (s_1,s_2).

Definition 1.5: The integral (1.5.2.2) is said to be absolutely convergent if

$$\lim_{\substack{a\to\infty \\ b\to\infty}} \int_0^a\int_0^b \left|\exp(-s_1x - s_2y)f(x,y)\right| dxdy = \int_0^\infty\int_0^\infty \exp(-s_1x - s_2y)\left|f(x,y)\right| dxdy$$

where $\Re e\, s_1 = \sigma$, $\Re e\, s_2 = \tau$.

Theorem 1.5. If the integral (1.5.2.2) converges absolutely at the point (s_{10}, s_{20}), then it converges absolutely at all points (s_1,s_2) for which $\Re e(s_1 - s_{10}) > 0$ and $\Re e(s_2 - s_{20}) > 0$.

Theorem 1.6. If the function $f(x,y)$ satisfies the inequality (1.5.2.3), then the integral (1.5.2.2) is absolutely convergent at all points (s_1,s_2) for which $\Re e\, s_1 > h$ and $\Re e\, s_2 > k$. Also

$$|F(s_1,s_2)| < \frac{M}{(\sigma-h)(\tau-k)} \quad \text{where } \Re e\, s_1 = \sigma \text{ and } \Re e\, s_2 = \tau.$$

Remark 4: Absolute convergence of the integral (1.5.2.1) at the point (s_{10}, s_{20}) implies bounded convergence at this point, and also at all point (s_1, s_2) for which $\Re e\,(s_1 - s_{10}) \geq 0$ and $\Re e(s_2 - s_{20}) \geq 0$.

Remark 5: The complete analysis of the convergence of the double Laplace transformation using the theory of Lebesgue integration is given in Ditkin and Prudnikov [43] and Coon and Bernstein [18] in great detail.

Theorem 1.7. The function $F(s_1, s_2)$ is analytic in the region D. Moreover,

$$\frac{\partial^{m+n}}{\partial s_1{}^m \partial s_2{}^n} F(s_1,s_2) = (-1)^{m+n} \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 y) x^m y^n f(x,y)\,dxdy, \quad (1.5.2.4)$$

and the function $x^m y^n f(x,y)$ belongs to the class Ω_2.

Theorem 1.8. In order that $f(x,y)$ shall belong to the class Ω_2, it is necessary and sufficient that the inequality

$$\left|\exp(-a\alpha - b\beta)\int_0^a \int_0^b f(x,y)\,dxdy\right| < M \quad (a \geq 0, b \geq 0) \qquad (1.5.2.5)$$

holds for one pair of values (α, β), $\alpha > 0, \beta > 0$.

Theorem 1.9. If the function $f(x,y)$ belongs to the class Ω_2, then the function

$$f_{u,v}(x,y) = \begin{cases} f(x-u, y-v) & \text{for } x > u,\ y > v \\ 0 & \text{otherwise} \end{cases} \qquad (u, v > 0)$$

also belongs to the class Ω_2, and

$$\int_0^\infty \exp(-s_1x)dx\int_0^\infty \exp(-s_2y)f_{u,v}(x,y)dy$$
$$= \exp(-ux-vy)\int_0^\infty \exp(-s_1x)dx\int_0^\infty \exp(-s_2y)f(x,y)dy.$$

Remark 6: The existence of the transform of a function in two variables does not imply the existence of transforms of its derivatives. However, if these transforms exist, then they can be found in the same way as in the one dimensional case. We list the following formulas for use in Chapter 4.

Remark 7: If

$$u(x,0) = f(x),\ u(0,y) = g(y),$$
$$u_y(x,y)\big|_{y=0} = u_y(x,0) = f_1(x),\ u_x(x,y)\big|_{x=0} = u_x(0,y) = g_1(y)$$

and if their one-dimensional Laplace transformations are $F(s_1), G(s_2), F_1(s_1)$ and $G_1(s_2)$, respectively, then

$$\mathcal{L}_2\{u(x,y);s_1,s_2\} = \int_0^\infty\int_0^\infty \exp(-s_1x-s_2y)u(x,y)dxdy = U(s_1,s_2) \tag{1.5.2.6}$$

$$\mathcal{L}_2\{u_x;s_1,s_2\} = s_1U(s_1,s_2) - G(s_2) \tag{1.5.2.7}$$

$$\mathcal{L}_2\{u_{xx};s_1,s_2\} = s_1^2U(s_1,s_2) - s_1G(s_2) - G_1(s_2) \tag{1.5.2.8}$$

$$\mathcal{L}_2\{u_y;s_1,s_2\} = s_2U(s_1,s_2) - F(s_1) \tag{1.5.2.9}$$

$$\mathcal{L}_2\{u_{yy};s_1,s_2\} = s_2^2U(s_1,s_2) - s_2F(s_1) - F_1(s_1) \tag{1.5.2.10}$$

$$\mathcal{L}_2\{u_{xy};s_1,s_2\} = s_1s_2U(s_1,s_2) - s_1F(s_1) - s_2G(s_2) + u(0,0) \tag{1.5.2.11}$$

<u>The Inversion Theorem</u>: Suppose that $f(x,y)$ has first order partial

derivatives $f_x(x,y), f_y(x,y)$ and mixed second order partial derivatives $f_{xy}(x,y)$ and that we can find positive constants Q, k_1, k_2 , such that for all $0 < x < \infty, 0 < y < \infty$

$$|f(x,y)| < Q\exp(k_1 x + k_2 y),\ |f_{xy}(x,y)| < Q\exp(k_1 x + k_2 y) \qquad (1.5.2.12)$$

Then, if

$$F(s_1,s_2) = \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 y) f(x,y)\,dxdy$$

we have

$$f(x,y) = \lim_{\substack{\omega_1\to\infty \\ \omega_2\to\infty}} \frac{1}{(2\pi i)^2} \int_{\sigma-i\omega_1}^{\sigma+i\omega_1} \int_{\tau-i\omega_2}^{\tau+i\omega_2} \exp(s_1 x + s_2 y) F(s_1,s_2)\,ds_1 ds_2 \qquad (1.5.2.13)$$

where $\sigma > k_1$ and $\tau > k_2$.

Definition 1.6: We define the following integral as Laplace-Carson transform

$$F(s_1,s_2) = s_1 s_2 \int_0^\infty \int_0^\infty \exp(-s_1 x - s_2 y) f(x,y)\,dxdy \qquad (1.5.2.14)$$

and we shall write

$$F(s_1,s_2) \overset{\bullet\bullet}{\underset{\bullet\bullet}{=}} f(x,y)$$

by analogy with one-dimensional symbolism

$$F(s) \overset{\bullet}{\underset{\bullet}{=}} f(x)$$

The function $F(s_1,s_2)$ is called the *image,* and the function $f(x,y)$ the *original.*

Remark 8: Because the Laplace-Carson transform differs from the Laplace transform only in the factor $s_1 s_2$, it is clear that all the theorems and properties of the Laplace transform can be reformulated for the Laplace -

Carson transform.

Some additional operational properties of the double Laplace transformation (1.5.2.2) are listed below for further reference in Chapter 4.

Suppose that $F(s_1,s_2)$ is the image of the function $f(x,y)$ and let

$$\frac{\partial F(s_1,s_2)}{\partial s_1}=F_{s_1}(s_1,s_2),\ \frac{\partial F(s_1,s_2)}{\partial s_2}=F_{s_2}(s_1,s_2),\ \frac{\partial F(s_1,s_2)}{\partial s_1\partial s_2}=F_{s_1s_2}(s_1,s_2),$$
$$\frac{\partial f(x,y)}{\partial x}=f_x(x,y),\ \frac{\partial f(x,y)}{\partial y}=f_y(x,y).$$

Then

$$\mathcal{L}_2\{-xf(x,y);s_1,s_2\}=F_{s_1}(s_1,s_2) \tag{1.5.2.15}$$

$$\mathcal{L}_2\{-yf(x,y);s_1,s_2\}=F_{s_2}(s_1,s_2) \tag{1.5.2.16}$$

$$\mathcal{L}_2\{xyf(x,y);s_1,s_2\}=F_{s_1s_2}(s_1,s_2)\ \text{ etc.} \tag{1.5.2.17}$$

$$\mathcal{L}_2\left\{\frac{1}{x+1}f(x,y);s_1,s_2\right\}=\int_{s_1}^{\infty}\exp[-(\lambda-s_1)F(\lambda,s_2)d\lambda \tag{1.5.2.18}$$

$$\mathcal{L}_2\left\{\frac{1}{x+y}f(x,y);s_1,s_2\right\}=\int_{s_1}^{\infty}F(\lambda,s_2+\lambda-s_1)d\lambda \tag{1.5.2.19}$$

$$\mathcal{L}_2\left\{\frac{1}{x}f(x,y);s_1,s_2\right\}=\int_{s_1}^{\infty}F(\lambda,s_2)d\lambda \tag{1.5.2.20}$$

$$\mathcal{L}_2\left\{\frac{1}{xy}f(x,y);s_1,s_2\right\}=\int_0^{\infty}\int_0^{\infty}F(\lambda,\mu)d\lambda d\mu \tag{1.5.2.21}$$

$$\mathcal{L}_2\left\{\frac{1}{a}f(\tfrac{x}{a},y);s_1,s_2\right\}=F(as_1,s_2),\quad a>0. \tag{1.5.2.22}$$

$$\mathcal{L}_2\{\exp(ax)f(x,y);s_1,s_2\}=F(s_1-a,s_2) \tag{1.5.2.23}$$

$$\mathcal{L}_2\{f(y,x);s_1,s_2\}=F(s_2,s_1) \tag{1.5.2.24}$$

$$\mathcal{L}_2\left\{\frac{1}{ab}f(\tfrac{x}{a},\tfrac{y}{b});s_1,s_2\right\}=F(as_1,bs_2),\quad a,b>0 \tag{1.5.2.25}$$

$$\mathcal{L}_2\{\exp(-\gamma x-\rho y)f(x,y);s_1,s_2\}=F(s_1+\gamma,s_2+\rho) \tag{1.5.2.26}$$

$$\mathcal{L}_2\left\{\int_0^x f(\xi,y)d\xi;s_1,s_2\right\}=\frac{1}{s_1}F(s_1,s_2) \tag{1.5.2.27}$$

Let $g(x,y)=\begin{cases}\int_0^x f(x-\xi,y-\xi)d\xi & \text{if } y>x\\ \int_0^y f(x-\xi,y-\xi)d\xi & \text{if } y<x\end{cases}$, then

$$\mathcal{L}_2\{g(x,y);s_1,s_2\}=\frac{1}{s_1+s_2}F(s_1,s_2). \tag{1.5.2.28}$$

$$\mathcal{L}_2\left\{\int_0^x f(x-\xi,y+\xi)d\xi;s_1,s_2\right\}=\frac{1}{s_1-s_2}F(s_1,s_2) \tag{1.5.2.29}$$

Let $g(x,y)=\begin{cases}\frac{1}{a}\int_0^x \exp(-\frac{\gamma}{a}\xi)f(x-\xi,y-\frac{b}{a}\xi)d\xi & \text{if } y>\frac{b}{a}x\\ \frac{1}{b}\int_0^y \exp(-\frac{\gamma}{b}\eta)f(x-\frac{a}{b}\eta,y-\eta)d\eta & \text{if } y<\frac{b}{a}x\end{cases}$, then

$$\mathcal{L}_2\{g(x,y);s_1,s_2\}=\frac{1}{as_1+bs_2+\gamma}F(s_1,s_2),\ \frac{b}{a}>0. \tag{1.5.2.30}$$

Let $g(x,y)=\begin{cases}f(x,y)-\int_0^x f_y(x-\xi,y-\xi)d\xi & \text{if } y>x\\ \int_0^\infty f_x(x-\xi,y-\xi)d\xi & \text{if } y<x\end{cases}$,then

$$\mathcal{L}_2\{g(x,y);s_1,s_2\}=\frac{s_1}{s_1+s_2}F(s_1,s_2). \tag{1.5.2.31}$$

$$\mathcal{L}_2\left\{\int_0^x\int_0^y f(\xi,\eta)d\xi d\eta;s_1,s_2\right\}=\frac{1}{s_1s_2}F(s_1,s_2) \tag{1.5.2.32}$$

$$\mathcal{L}\left\{\int_0^x f(x-\xi,y-\eta)\sin\xi d\xi;s_1,s_2\right\}=\frac{1}{(s_1+s_2)^2+1}F(s_1,s_2) \tag{1.5.2.33}$$

$$\mathcal{L}_2\left\{\int_0^y f(x-\xi,y-\eta)\cos\xi d\xi;s_1,s_2\right\}=\frac{s_1+s_2}{(s_1+s_2)^2+1}F(s_1,s_2) \tag{1.5.2.34}$$

For details we refer to Ditkin and Prudnikov [43] and Voelker and Doetsch [107].

1.6. Recapitulations on N-dimensional Laplace Transformations

1.6.1. Definition and Basic Properties

As *original functions*, shortly called *originals* , of the Laplace transformation consider the elements of the following space of functions:

Definition 1.7: Let $E_{\overline{a}}$, $\overline{a}\in\mathbf{R}^n$ be the set of functions from $\mathbf{R}^n$ into $\mathbf{C}$ with the following properties:

There exists a point $\bar{a} \in \mathrm{R}^n$ such that $f \in L_1[\mathbf{R}^n_+; \exp(-\bar{a}.\bar{x})]$ and

$$f(\bar{x}) = 0, \quad \bar{x} \in \mathrm{R}^n_+ \setminus \overline{\mathrm{R}^n_+}, \tag{1.6.1.1}$$

That is, if at least one component x_j of $\bar{x}$ is negative.

$E_{\bar{a}}$ is equipped with the norm $\|f\|_{E_{\bar{a}}} = \|f\|_{L_1[\mathbf{R}^n_+;\exp(-\bar{a}.\bar{x})]}$.

Remark 1: Because of property (1.6.1.1) the originals are sometimes written by means of the n-dimensional Heaviside function θ_n in the form

$$f(\bar{x}) = \theta_n(\bar{x}) f_1(\bar{x}), \quad \bar{x} \in \mathrm{R}^n$$

we usually omit the factor θ_n and we assume that originals f have the property (1.6.1.1).

Definition 1.8: The n-dimensional Laplace transform of a function from R^n_+ into C is defined by means of

$$\begin{aligned} F(\bar{s}) = \mathrm{L}_n\{f(\bar{x}); \bar{s}\} &= \int_0^\infty \int_0^\infty \ldots \int_0^\infty \exp[-\bar{s}.\bar{x}] f(\bar{x}) dx_1 dx_2 \ldots dx_n \\ &= \int_{\mathbf{R}^n_+} \exp[-\bar{s}.\bar{x}] f(\bar{x}) d\bar{x} \end{aligned} \tag{1.6.1.2}$$

The domain of definition of F is the set of all points $\bar{s} \in \mathrm{C}^n$ such that the integral in (1.6.1.2) is convergent.

Theorem 1.10. Let $f \in E_{\bar{a}}$ and $H_{\bar{a}} = \{\bar{s} : \bar{s} \in \mathrm{C}^n, \ \Re e\,\bar{s} > \bar{a}\}$, $\overline{H_{\bar{a}}} = \{\bar{s} : \bar{s} \in \mathrm{C}^n, \ \Re e\,\bar{s} \geq \bar{a}\}$. Then the Laplace integral (1.6.1.2) is absolutely and uniformly convergent on $\overline{H_{\bar{a}}}$. F

is an analytic function on $H_{\bar{a}}$ and it holds

$$D^{k}F(\bar{s})=(-1)^{p_1(\bar{k})}\mathcal{L}_n\left\{\bar{x}^{\bar{k}}f(\bar{x});\bar{s}\right\} \tag{1.6.1.3}$$

where $\bar{k}=(k_1,k_2,\dots,k_n)\in\mathbf{N}_0^n$.

Corollary 1.1. Let $f\in E_{\bar{a}}$ and let the Laplace transform F of f be convergent at a point $\bar{s}_0\in H_{\bar{a}}$. Then it also converges at $\overline{H}_{\bar{\sigma}_0}=\left\{\bar{s}:\bar{s}\in\mathbf{C}^n,\ \Re e\,\bar{s}\geq\Re e\,\bar{s}_0=\bar{\sigma}_0\right\}$.

Theorem 1.11. Let $f,g\in E_{\bar{a}}$ and $\alpha,\beta\in\mathbf{C}$. Then $\alpha f+\beta g\in E_{\bar{a}}$ and

$$\mathcal{L}_n\{\alpha f+\beta g;\bar{s}\}=\alpha\mathcal{L}_n\{f:\bar{s}\}+\beta\mathcal{L}_n\{g:\bar{s}\}.$$

Remarks:

1. Instead of $E_{\bar{a}}$ one sometimes considers a space $\tilde{E}_{\bar{a}}$ of functions of $L_1^{Loc}(\mathbf{R}_+^n)$ with the property (1.6.1.1) and that there exists a point $\bar{a}\in\mathbf{R}^n$ and a positive number M such that

$$\left|f(\bar{x})\right|\leq M\exp(\bar{a}.\bar{x}),\ \ \bar{x}\geq\overline{X}$$

Then for every $\bar{\varepsilon}\in\mathbf{R}_+^n$ obviously $\tilde{E}_{\bar{a}}\subset E_{\bar{a}+\bar{\varepsilon}}$. Therefore the statements of Theorem 1.7 remains true, if $\overline{H}_{\bar{a}}$ is replaced by $\overline{H}_{\bar{a}+\bar{\varepsilon}}$, $\bar{\varepsilon}\in\mathbf{R}_+^n$, arbitrary. The *image* $F=\mathcal{L}_n\{f:\bar{s}\}$ are again analytic on $H_{\bar{a}}$.

2. Even less restrictive conditions for originals are:

(a) There exists a point $\bar{a}\in\mathbf{C}^n$ such that $\int\limits_{\mathbf{R}_+^n(\bar{v})}\exp(-\bar{a}.\bar{x})d\bar{x}$ is convergent for every $\bar{v}\in\mathbf{R}_+^n$,

(b) These integrals are uniformly bounded with respect to $\bar{v}\in\mathbf{R}_+^n$, i.e.

$$\left|\int\limits_{\mathbf{R}_+^n(\bar{v})}\exp(-\bar{a}.\bar{x})f(\bar{x})d\bar{x}\right|\leq M(\bar{a}),$$

where $M(\bar{a})$ is independent of $\bar{v}$, and

(c) There exists $\lim\limits_{\bar{v}\to\infty} \int\limits_{\mathbf{R}^n_+(\bar{v})} \exp(-\bar{a}.\bar{x}) f(\bar{x}) d\bar{x}$.

This convergence is called bounded convergence of the Laplace integral (1.6.1.2). Then one can prove $H_{\bar{a}}$ to be a domain of absolute convergence of the Laplace integral (1.6.1.2) and again the Laplace transform is an analytic function on this domain. For details we refer to Voelker and Doetsch [107] and Ditkin and Prudnikov [43]. Obviously from $f \in E_{\bar{a}}$ it follows that $L_n\{f;\bar{s}\}$ is boundedly convergent on $\overline{H}_{\bar{a}}$ and from $f \in \tilde{E}_{\bar{a}}$ it follows that $L_n\{f;\bar{s}\}$ is boundedly convergent on $H_{\bar{a}}$.

3. Instead of the N-dimensional Laplace transform (1.6.1.2) one is sometimes calculating the so - called N-dimensional Laplace–Carson transform:

$$F(\bar{s}) = p_n(\bar{s}) \int\limits_{\mathbf{R}^n_+} \exp(-\bar{s}.\bar{x}) f(\bar{x}) d\bar{x} \tag{1.6.1.4}$$

Symbolically we denote the pair $F(\bar{s})$ and $f(\bar{x})$ with the operational relation

$$F(\bar{s}) \overset{n}{\underset{n}{=}} f(\bar{x}) \quad \text{or} \quad f(\bar{x}) \overset{n}{\underset{n}{=}} F(\bar{s}).$$

In this notation some formulas become more simple.

Theorem 1.12. Let $f \in E_{\bar{a}}$ and $F(\bar{s}) = L_n\{f(\bar{x});\bar{s}\}$. Then

$$\lim_{\sigma_j\to\infty} F(\bar{s}) = 0, \ j = 1,2,\ldots,n \ , \sigma_j = \Re e \ s_j. \tag{1.6.1.5}$$

From (1.6.1.5) we get immediately

Corollary 1.2. If $\bar{s} \in H_{\bar{a}}$, and $|\bar{s}|$ tends to $+\infty$ such that at least one of the quantities $\sigma_j = \Re e\, s_j, j \in \{1,2,\dots,n\}$ tends to $+\infty$, then

$$\lim_{|\bar{s}| \to +\infty} F(\bar{s}) = 0,$$

uniformly with respect to $\operatorname{Im} \bar{s} = \bar{\tau}$.

Remark 4: Condition (1.6.1.5) is a necessary condition for a function F, analytic in $H_{\bar{a}}$, to be the Laplace transform of an original from $E_{\bar{a}}$. So, for example, the functions $\sin \bar{s}$ and $\bar{s}^{\bar{\alpha}}$, $\bar{\alpha} \in \mathbf{R}^n \setminus \mathbf{R}_+^n$ can not be Laplace transforms.

As $\bar{\tau} = \operatorname{Im} \bar{s}$ tends to $\pm\infty$, the asymptotic behavior is given by

Theorem 1.13. If $f \in E_{\bar{a}}$, ,then

$$\lim_{\tau_j \to \pm\infty} F(\bar{\sigma}_j + i\bar{\tau}_j) = 0, \quad \bar{\sigma} \geq \bar{a}, \quad j = 1,2,\dots,n. \tag{1.6.1.6}$$

So for example $\exp(-p_1(\bar{s}))$ can not be a Laplace transform, though it tends to zero, if $\Re e\, s_j$ tends to $\pm\infty$, since it does not tends to zero, if $\operatorname{Im} \sigma_j$ tends to $\pm\infty (\Re e\, s_j > 0, \text{fixed})$.

Theorem 1.14. If a function F, analytic on $H_{\bar{a}}$, should be an (absolutely convergent) Laplace transform, then it must necessarily be

$$\lim_{s_j \to \infty} F(\bar{s}) = 0, \quad j = 1,2,\dots,n,$$

if every s_j tends to infinity within $\overline{H}_{\sigma_j}$, $\sigma_j > a_j$, the other variables s_k, $k \neq j$ being fixed $(\Re e(s_k) > a_k)$.

In the above explanation we derived some necessary conditions for a function F to be an (absolutely convergent) Laplace transform. Sufficient conditions are difficult to formulate. For details we refer to Brychkov et al. [11, ch. 2].

1.6.2. The Inversion of the Laplace Transformation

Theorem 1.15. Let $f \in E_{\overline{a}}$, $\overline{a} \in \mathbf{R}_+^n$ and let there exist $\frac{\partial}{\partial x_j} f = D_j f$, $j = 1,2,\ldots,n$, $D^1 f = \frac{\partial^n f}{\partial x_1 \partial x_2 \ldots \partial x_n}$ and $D_j f \in C(\mathbf{R}_+^n)$, $D^1 f \in E_{\overline{a}}$. Then at each point of continuity of f the so called complex inversion formula holds

$$\mathfrak{L}_n^{-1}\{F(\overline{s});\overline{x}\} = f(\overline{x}) = (2\pi i)^{-n} \int_{(\overline{\alpha})} \exp(\overline{s}.\overline{x}) F(\overline{s}) d\overline{s}, \ \overline{\alpha} \in \mathbf{R}_+^n, \ \overline{\alpha} > a. \quad (1.6.2.1)$$

Here the integral has to be understood in the sense of the principal value of Cauchy, i.e.

$$\int_{(\overline{\alpha})} \ldots d\overline{s} = \lim_{\substack{\beta_j \to \infty \\ j=1,2,\ldots,n}} \int_{\alpha_1 - i\beta_1}^{\alpha_1 + i\beta_1} \int_{\alpha_2 - i\beta_2}^{\alpha_2 + i\beta_2} \ldots \int_{\alpha_n - i\beta_n}^{\alpha_n + i\beta_n} \ldots ds_1 ds_2 \ldots ds_n \qquad (1.6.2.2)$$

Definition 1.9: The integral at the right-side of (1.6.2.1)) is called the n-dimensional inverse Laplace transformation of F.

Remark 1: In particular, if $F(\overline{s}) = \prod_{j=1}^{n} F_j(s_j)$, then from (1.6.2.1) we obtain

$$f(\overline{x}) = \prod_{j=1}^{n} f_j(t_j),$$

where $f_j = \mathfrak{L}_n^{-1}\{F_j;x_j\}$.

Theorem 1.16. Let $f \in E_{\bar{a}}$ and $F(\bar{s}) = \mathfrak{L}_n\{f(\bar{x});\bar{s}\}$. Then at each point of continuity of f we have (1.6.2.1), (1.6.2.2), where the limit has to be understood in the sense of *distributions* of the space $D'(\mathbf{R}^n)$. For details we refer to Brychkov et al. [11; secs. 1.3.1 and 2.3.1].

Theorem 1.17. Let $f \in E_{\bar{a}}$, $\bar{\alpha} > \bar{a}$. In addition, if $F(\bar{\alpha}+i\bar{\tau}) = \mathfrak{L}_n\{f(\bar{x});\bar{\alpha}+i\bar{\tau}\}$ belongs to L_1 with respect to $\bar{\tau}$, then inversion formula (1.6.2.1) holds (a.e).

Theorem 1.18. Let F be analytic on $H_{\bar{a}}$ and $\lim\limits_{|s_j|\to\infty} F(\bar{s}) = 0$, if $\Re e\, s_j \geq \alpha_j > a_j$, $j = 1,2,\ldots,n$, uniformly with respect to $\arg(s_j)$. Furthermore, let $\int\limits_{(\bar{\alpha})} |F(\bar{s})|d\bar{s}$ exists for every $\bar{\alpha} > \bar{a}$, i.e. $F \in L_1(\bar{\alpha})$. Then F is a Laplace transform of a function $f \in E_{\bar{a}+\bar{\varepsilon}}$, $\bar{\varepsilon} \in \mathbf{R}_+^n$, arbitrary, which is a.e. continuous and f is defined by the complex inversion formula (2.1).

Remark 2: The inversion formula of the Laplace transform is of more or less theoretical interest. The complex inversion formula (1.6.2.1) may be of interest if the integral can be calculated by the method of residues in the one-dimensional case.

In proving theorems in Chapters 2 and 3, we use the process of the change in the order of integration of multiple integrals and their conversion to repeated integrals. For the justification of this process we use the following theorem due to Fubini as, given by Brychkov et al. [11, p. 8].

Theorem 1.19. (Fubini). If $f(\bar{x},\bar{y}) \in L_1(\mathbf{R}^n \times \mathbf{R}^m)$, then for a.e. $\bar{x} \in \mathbf{R}^n$ we have $f(\bar{x},\cdot) \in L_1(\mathbf{R}^m)$ and for a.e. $\bar{y} \in \mathbf{R}^m$, $f(\cdot,\bar{y}) \in L_1(\mathbf{R}^n)$. Furthermore, there exist the integrals $\int\limits_{\mathbf{R}^m} f(\bar{x},\bar{y})d\bar{y} \in L_1(\mathbf{R}^n)$ and $\int\limits_{\mathbf{R}^n} f(\bar{x},\bar{y})d\bar{x} \in L_1(\mathbf{R}^m)$ and we have

$$\int\limits_{\mathbf{R}^{m+n}} f(\bar{x},\bar{y})d\bar{x}d\bar{y} = \int\limits_{\mathbf{R}^m}\left(\int\limits_{\mathbf{R}^n} f(\bar{x},\bar{y})d\bar{x}\right)d\bar{y} = \int\limits_{\mathbf{R}^n}\left(\int\limits_{\mathbf{R}^m} f(\bar{x},\bar{y})d\bar{y}\right)d\bar{x}.$$

CHAPTER 2. THEOREMS REGARDING N-DIMENSIONAL LAPLACE TRANSFORMATIONS AND THEIR APPLICATIONS

In this chapter, we will derive a number of theorems and corollaries for calculating Laplace transform and inverse Laplace transform pairs of N-dimensions from known one-dimensional Laplace transformations. Regarding their application, we provide several examples demonstrating ways of obtaining new N-dimensional transformation pairs.

This chapter consists of five theorems in two sections. Section one is divided in two parts. Part one deals with original functions of the form $F[2p_1(\overline{x^{-1}})]$ and describes our formulation of a theorem the involving the Laplace transform of functions of the form

$$p_n(\overline{x^{-\frac{1}{2}}})\, F_j[2p_1(\overline{x^{-1}})], \text{ where } j = 0, 1, 2.$$

Part two is concerned with image functions with arguments $[p_1(\overline{s^{\frac{1}{2}}})]^2$. In this part, we have formulated theorems involving the inverse Laplace transforms of functions of the form

$$\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2],\ \frac{[p_1(\overline{s^{\frac{1}{2}}})]^5}{p_n(\overline{s^{\frac{1}{2}}})}\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2], \text{ and } \frac{\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]}{p_n(\overline{s^{\frac{1}{2}}})}.$$

The next section contains theorems which deal with the inverse Laplace transforms of functions in the form

$$\frac{\gamma[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}]}{[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}} \text{ and } \frac{\Lambda[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}]}{[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}.$$

The theorems in this chapter are developed according to an idea obtained from papers of R. S. Dahiya[21] and [30].

2.1. Theorems, Corollaries and Applications Regarding N-dimensional Laplace Transformations

2.1.1. The Image of Functions with the Argument of $2p_1(\overline{x^{-1}})$ (Theorem 2.1.1 and Theorem 2.1.2)

This section begins with Theorem 2.1.1, which has three parts. We present the proof of only the first part in detail; the proofs of two other parts are provided in brief because all three are based upon similar ideas.

Theorem 2.1.1.

Let **(*i*)** $\mathbf{L}\{f(x); s\} = \phi(s)$, and

(*ii*) $\mathbf{L}\{x^{\frac{j-1}{2}} f(x^2); s\} = F_j(s)$ for $j = 0, 1, 2,$

assuming that $f(x^2)$ is a function of class Ω.

(a) If

(a1) $\mathbf{L}\{x^{-\frac{3}{2}} \phi(\frac{1}{x}); s\} = \xi(s)$,

(a2) $\mathbf{L}\{x^{-\frac{3}{2}} \xi(\frac{1}{x^2}); s\} = \eta(s)$,

where $x^{-\frac{3}{2}} \phi(\frac{1}{x})$ and $x^{-\frac{3}{2}} \xi(\frac{1}{x^2})$ are also functions of class Ω, and $x^{-\frac{3}{2}} \exp(-sx - \frac{u}{x}) f(u)$ and $u^{-\frac{1}{2}} x^{-\frac{3}{2}} \exp(-sx - \frac{2u^{\frac{1}{2}}}{x}) f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$.

Then

$$\mathcal{L}_n\left\{\frac{1}{p_n(x^{\frac{1}{2}})}F_o[2p_1(\overline{x^{-1}})];\bar{s}\right\} = \frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}\,p_n(s^{\frac{1}{2}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2], \tag{1.1.1}$$

where n = 2, 3, ..., N and $\Re e[p_1(\overline{s^{\frac{1}{2}}})] > c$ and c is a constant. It is assumed that the integrals involved exist. The existence conditions for two-dimensions are given in Ditkin and Prudnikov [43; p. 4] and similar conditions hold for N-dimensions, we refer to Brychkov et al. [11;ch.2].

(b) Assume the conditions (*i*), (*ii*) and (a1), and replace (a2) by

(b1) $\mathbf{L}\{x^{-\frac{1}{2}}\xi(\frac{1}{x^2});s\} = \zeta(s).$

Then

$$\mathcal{L}_n\left\{\frac{1}{p_n(x^{\frac{1}{2}})}F_1[2p_1(\overline{x^{-1}})];\bar{s}\right\} = \frac{\pi^{\frac{n-2}{2}}}{2}\cdot\frac{p_1(\overline{s^{\frac{1}{2}}})}{p_n(s^{\frac{1}{2}})}\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2], \tag{1.1.2}$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})] > d$, a constant, provided the integrals involved exist for

n = 2, 3, ..., N.

(c) Suppose that the conditions (*i*) and (*ii*) hold. If

(c1) $\mathbf{L}\{x^{-\frac{1}{2}}\phi(\frac{1}{x});s\} = \theta(s),$

(c2) $\mathbf{L}\{x^{-\frac{5}{2}}\theta(\frac{1}{x^2});s\} = \delta(s).$

Then

$$\mathcal{L}_n\left\{\frac{1}{p_n(x^{\frac{1}{2}})}F_2[2\overline{p_1(x^{-1})}];\bar{s}\right\} = \frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}p_n(s^{\frac{1}{2}})}\delta[(\overline{p_1(s^{\frac{1}{2}})})^2], \tag{1.1.3}$$

provided that $\Re e[\overline{p_1(s^{\frac{1}{2}})}] > e$, where e is a constant and where $x^{-\frac{5}{2}}\theta(\frac{1}{x^2})$ is a function of class Ω. Assume that the integrals involved exist for n = 2, 3, ..., N.

Proof (a): From (*i*), we obtain $\phi(s) = \int_0^\infty \exp(-st)f(t)dt$ for $\Re e\ s > c_0$, where c_0 is a constant, so that

$$x^{-\frac{3}{2}}\phi(\frac{1}{x}) = x^{-\frac{3}{2}}\int_0^\infty \exp(-\frac{u}{x})f(u)du. \tag{1.1.4}$$

Multiplying both sides of (I.1.4) by exp(-*sx*) and integrating from 0 to ∞, we obtain

$$\int_0^\infty \exp(-sx)x^{-\frac{3}{2}}\phi(\frac{1}{x})dx = \int_0^\infty [\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})f(u)du]dx.$$

The integrand $x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, that, by Fubini's Theorem, interchanging the order of the integral on the right of (1.1.4) is permissible. By using (a1) on the left and interchanging the order of integration on the right side of (1.1.4), we obtain

$$\xi(s) = \int_0^\infty f(u)[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})dx]du, \text{ where } \Re e\ s > \lambda_1 \text{ and } \lambda_1 \text{ is a constant.} \tag{1.1.5}$$

A result in Roberts and Kaufman [87] regarding the inner integral in (1.1.5) be used to evaluate this integral as

$$\int_0^\infty x^{-\frac{3}{2}} \exp(-sx - \frac{u}{x})dx = \int_0^\infty x^{-\frac{3}{2}} \exp(-sx - \frac{(2u^{\frac{1}{2}})^2}{4x})dx$$
$$= \pi^{\frac{1}{2}} u^{-\frac{1}{2}} \exp(-2u^{\frac{1}{2}} s^{\frac{1}{2}}).$$

Therefore, (1.1.5) can be written as

$$\xi(s) = \pi^{\frac{1}{2}} \int_0^\infty u^{-\frac{1}{2}} \exp(-2u^{\frac{1}{2}} s^{\frac{1}{2}}) f(u)du. \tag{1.1.6}$$

Using (1.1.6) in (a2), we arrive at

$$\begin{aligned} \eta(s) &= \int_0^\infty x^{-\frac{3}{2}} \exp(-sx) \xi(\frac{1}{x^2})dx \\ &= \pi^{\frac{1}{2}} \int_0^\infty \exp(-sx) x^{-\frac{3}{2}} [\int_0^\infty u^{-\frac{1}{2}} \exp(-\frac{2u^{\frac{1}{2}}}{x}) f(u)du]dx \qquad (1.1.7) \\ &= \pi^{\frac{1}{2}} \int_0^\infty [\int_0^\infty x^{-\frac{3}{2}} u^{-\frac{1}{2}} \exp(-sx - \frac{2u^{\frac{1}{2}}}{x}) f(u)du]dx, \end{aligned}$$

where $\Re e\ s > \lambda_2$ for some constant λ_2.

Again, because $x^{-\frac{3}{2}} u^{-\frac{1}{2}} \exp(-sx - \frac{2u^{\frac{1}{2}}}{x}) f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, by Fubini's Theorem, we can interchange the order of integration on the right side of (1.1.7) to obtain

$$\eta(s) = \pi^{\frac{1}{2}} \int_0^\infty u^{-\frac{1}{2}} f(u) [\int_0^\infty x^{-\frac{3}{2}} \exp(-sx - \frac{(2^{\frac{3}{2}} u^{\frac{1}{4}})^2}{4x})dx]du. \tag{1.1.8}$$

Using the previously mentioned result in Roberts and Kaufman [87] on the inner integral in (1.1.5), we can evaluate the integral inside the brackets. Consequently, equation (1.1.8) becomes

$$\eta(s)=2^{-\frac{1}{2}}\pi\int_0^\infty u^{-\frac{3}{4}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du. \tag{1.1.9}$$

By substituting $u = \upsilon^2$ in (1.1.9), we obtain

$$\eta(s)=2^{\frac{1}{2}}\pi\int_0^\infty \upsilon^{-\frac{1}{2}}f(\upsilon^2)\exp(-2^{\frac{3}{2}}\upsilon^{\frac{1}{2}}s^{\frac{1}{2}})d\upsilon. \tag{1.1.10}$$

Replacing s by $[p_1(\overline{s^{\frac{1}{2}}})]^2$ and multiplying both sides of (1.1.10) by $p_n(\overline{s^{\frac{1}{2}}})$, we arrive at

$$p_n(\overline{s^{\frac{1}{2}}})\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2]=2^{\frac{1}{2}}\pi\int_0^\infty \upsilon^{-\frac{1}{2}}f(\upsilon^2)p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}\upsilon^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))d\upsilon. \tag{1.1.11}$$

Now we use the operational relation given in Ditkin and Prudnikov [43]:

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}}) \doteqdot (\pi x_i)^{-\frac{1}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, \mathrm{n} \tag{1.1.12}$$

Equation (1.1.11) can be rewritten as

$$p_n(\overline{s^{\frac{1}{2}}})\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \underset{n}{\overset{n}{=}} \frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_0^\infty \upsilon^{-\frac{1}{2}}f(\upsilon^2)\exp(-2\upsilon p_1(\overline{x^{-1}}))d\upsilon \tag{1.1.13}$$

Using (*ii*) for $j = 0$ from (1.1.13), we obtain

$$p_n(\overline{s^{\frac{1}{2}}})\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \underset{n}{\overset{n}{=}} \frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}}\cdot\frac{1}{p_n(\overline{x^{\frac{1}{2}}})}F_0[2p_1(\overline{x^{-1}})].$$

Hence,

$$\mathcal{L}_{\mathrm{n}}\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})}F_0[2p_1(\overline{x^{-1}})];\bar{s}\right\} = \frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}p_n(\overline{s^{\frac{1}{2}}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2],$$

where n = 2, 3, ..., N.

Proof (b): By our using (1.1.6) and (b1), it follows that

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty\Big[\int_0^\infty x^{-\frac{1}{2}}u^{-\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)du\Big]dx., \text{ where } \Re e\ s>\lambda_1. \qquad (1.1.14)$$

Clearly, $x^{-\frac{1}{2}}u^{-\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})$ belongs $L_1\,[(0,\infty)\times(0,\infty)]$; therefore, according to Fubini's Theorem, (1.1.14) can be rewritten as

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}f(u)\Big[\int_0^\infty x^{-\frac{1}{2}}\exp(-sx-\frac{(2^{\frac{3}{2}}u^{\frac{1}{4}})^2}{4x})dx\Big]du, \text{ where } \Re e\ s>\lambda_1.$$

From the tables of Roberts and Kaufman [87], we obtain

$$s^{\frac{1}{2}}\zeta(s)=\pi\int_0^\infty u^{-\frac{1}{2}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du. \qquad (1.1.15)$$

Next, we substitute $u=v^2$ in (1.1.15) to obtain

$$s^{\frac{1}{2}}\zeta(s)=2\pi\int_0^\infty \exp(-2^{\frac{3}{2}}s^{\frac{1}{2}}v^{\frac{1}{2}})f(v^2)dv. \qquad (1.1.16)$$

Replacing s by $[p_1(\overline{s^{\frac{1}{2}}})]^2$, multiplying both sides of (1.1.16) by $p_n(\overline{s^{\frac{1}{2}}})$, and using (1.1.12), equation (1.1.16) reads as follows

$$p_n(\overline{s^{\frac{1}{2}}})\,p_1(\overline{s^{\frac{1}{2}}})\,\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \overset{n}{\underset{n}{=}} \frac{2}{\pi^{\frac{n-2}{2}}\,p_n(\overline{x^{\frac{1}{2}}})}\int_0^\infty \exp\Big(-2\,v p_1(\overline{x^{-1}})\Big)f(v^2)dv \qquad (1.1.17)$$

Applying (ii) for $j=1$ in (1.1.17), we arrive at

$$p_n(\overline{s^{\frac{1}{2}}})\,p_1(\overline{s^{\frac{1}{2}}})\,\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \overset{n}{\underset{n}{=}} \frac{2}{\pi^{\frac{n-2}{2}}\,p_n(\overline{x^{\frac{1}{2}}})}F_1[2p_1(\overline{x^{-1}})].$$

Therefore,

$$\mathcal{L}_n\left\{\frac{1}{\overline{p_n(x^{\frac{1}{2}})}}F_1[2\overline{p_1(x^{-1})}];\bar{s}\right\}=\frac{\pi^{\frac{n-2}{2}}}{2}\cdot\frac{\overline{p_1(s^{\frac{1}{2}})}}{\overline{p_n(s^{\frac{1}{2}})}}\zeta[(\overline{p_1(s^{\frac{1}{2}})})^2],$$

where n = 2, 3, ..., N.

Proof (c): Following the same procedure as in parts (a) and (b), from (c1) and (c2), we obtain

$$\delta(s)=2^{-\frac{1}{2}}\pi\int_0^\infty u^{-\frac{1}{4}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du,\ \text{where } \Re e\ s>\lambda_3.$$

Now, we substitute $u = \upsilon^2$ on the right side to obtain

$$\delta(s)=2^{\frac{1}{2}}\pi\int_0^\infty \upsilon^{\frac{1}{2}}f(\upsilon^2)\exp(-2^{\frac{3}{2}}\upsilon^{\frac{1}{2}}s^{\frac{1}{2}})d\upsilon,\ \text{where } \Re e\ s>\lambda_3. \tag{1.1.18}$$

replacing s by $[\overline{p_1(s^{\frac{1}{2}})}]^2$, multiplying both sides of (1.1.18) by $\overline{p_n(s^{\frac{1}{2}})}$, and using (1.1.12), we can arrive at

$$\overline{p_n(s^{\frac{1}{2}})}\delta[(\overline{p_1(s^{\frac{1}{2}})})^2]\overset{n}{\underset{n}{=}}\frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}\overline{p_n(x^{\frac{1}{2}})}}\int_0^\infty \upsilon^{\frac{1}{2}}f(\upsilon^2)\exp(-2\upsilon\overline{p_1(x^{-1})})d\upsilon. \tag{1.1.19}$$

If we plug (*ii*) for $j = 2$ in (1.1.19), we obtain

$$\overline{p_n(s^{\frac{1}{2}})}\delta[(\overline{p_1(s^{\frac{1}{2}})})^2]\overset{n}{\underset{n}{=}}\frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}\overline{p_n(x^{\frac{1}{2}})}}F_2[2\overline{p_1(x^{-1})}].$$

Hence,

$$\mathcal{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})}F_2[2p_1(\overline{x^{-1}})];\bar{x}\right\}=\frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}p_n(\overline{s^{\frac{1}{2}}})}\delta[p_1(\overline{s^{\frac{1}{2}}})],$$

where n = 2, 3, ..., N.

Therefore, the theorem is proved.

2.1.1.1. Applications of Theorem 2.1.1

To show the applicability of Theorem 2.1.1, we will construct certain functions with n variables and calculate their Laplace transformation.

Example 1.1. Let $f(x)=x^{\frac{\upsilon}{4}}$. Then

$$\phi(s)=\frac{\Gamma(\frac{\upsilon}{4}+1)}{s^{\frac{\upsilon}{4}+1}},\ \Re e\ s>0,\ \Re e\ \upsilon>-4;$$

$$F_j(s)=\mathcal{L}\left\{x^{\frac{j-1}{2}}\cdot x^{\frac{\upsilon}{2}};s\right\}=\frac{\Gamma(\frac{j+\upsilon+1}{2})}{s^{\frac{j+\upsilon+1}{2}}}\ \text{for}\ j=0,1,2;$$

$$\xi(s)=\frac{\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{1}{2})}{s^{\frac{\upsilon}{4}+\frac{1}{2}}},\ \Re e\ \upsilon>-2,\ \Re e\ s>0,\ \text{and}$$

$$\eta(s)=\frac{\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{1}{2})\Gamma(\frac{\upsilon+1}{2})}{s^{\frac{\upsilon+1}{2}}},\ \Re e\ \upsilon>-1,\ \Re e\ s>0.$$

Therefore,

(a)

$$\mathcal{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{\upsilon+1}{2}}};\bar{s}\right\}=2^{\frac{\upsilon}{2}}\pi^{\frac{n-2}{2}}\Gamma(\tfrac{\upsilon}{4}+1)\Gamma(\tfrac{\upsilon}{4}+\tfrac{1}{2})\cdot\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+1}}$$

By the Duplication Theorem

$$= \pi^{\frac{n-1}{2}}\Gamma(\tfrac{\upsilon}{2}+1)\cdot\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+1}}, \tag{1.1}$$

where $\Re e\ \upsilon > -1$, $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})] > 0$, and $n = 2,\ 3,\cdots,\ N$.

Also,

$$\zeta(s) = \frac{\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{1}{2})\Gamma(\frac{\upsilon+3}{2})}{s^{\frac{\upsilon+3}{2}}},\ \Re e\ \upsilon > -2,\ \Re e\ s > 0.$$

Hence

(b)

$$\mathcal{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{\upsilon+2}{2}}};\ \bar{s}\right\} = \frac{\pi^{\frac{n-1}{2}}2^{\frac{\upsilon}{2}}\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{1}{2})\Gamma(\frac{\upsilon+3}{2})}{\Gamma(\frac{\upsilon+2}{2})}\cdot\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+2}}$$

By the Duplication Theorem

$$= \pi^{\frac{n-1}{2}}\Gamma(\tfrac{\upsilon}{2}+\tfrac{3}{2})\cdot\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+2}}, \tag{1.2}$$

where $\Re e\ \upsilon > -2$, $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})] > 0$, and $n = 2,\ 3,\cdots,\ N$.

Furthermore,

$$\theta(s) = \frac{\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{3}{2})}{s^{\frac{\upsilon}{4}+\frac{3}{2}}},\ \Re e\ \upsilon > -4,\ \Re e\ s > 0, \text{ and}$$

$$\delta(s) = \frac{\Gamma(\frac{\upsilon}{4}+1)\Gamma(\frac{\upsilon}{4}+\frac{3}{2})\Gamma(\frac{\upsilon}{2}+\frac{3}{2})}{s^{\frac{\upsilon}{2}+\frac{3}{2}}},\ \Re e\ \upsilon > -3,\ \Re e\ s > 0$$

so that

$$\mathcal{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{\upsilon+3}{2}}};\ \bar{s}\right\} = \pi^{\frac{n-1}{2}}\Gamma(\tfrac{\upsilon}{2}+2)\cdot\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+3}}, \tag{1.3}$$

where $\Re e\ \upsilon > -3$, $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})] > 0$, and $n = 2,\ 3,\cdots,\ N$.

Remark 2.1.1.1: Actually, the formulas (1.2) and (1.3) can be derived from formula (1.1) by replacing υ with $\upsilon + 1$ and $\upsilon + 2$, respectively. However, for many other examples these three formulas are different. The validity of the claim is illustrated by the following examples.

Example 1.2. Assume that $f(x) = \sin(ax^{\frac{1}{2}})$. Then

$$\phi(s) = \frac{a\pi^{\frac{1}{2}}}{2} s^{-\frac{3}{2}} \exp(-\frac{a^2}{4s}),\ \Re e\, s > 0, \text{ and}$$

$$F_j(s) = \frac{\Gamma(\frac{j+1}{2})}{(s^2+a^2)^{\frac{j+1}{4}}} \sin\,[\frac{j+1}{4}\tan^{-1}(\frac{a}{s})] \text{ for } j = 0,\ 1,\ 2 \text{ and } \Re e\, s > |\operatorname{Im} a|.$$

$$\xi(s) = \frac{2a\pi^{\frac{1}{2}}}{4s+a^2},\ \Re e\, s > -\Re e\frac{a^2}{4}. \tag{2.1}$$

$$\eta(s) = \frac{\pi}{a} s^{\frac{1}{2}} S_{-1,\frac{1}{2}}(\frac{2s}{a}),\ \Re e\, \mathrm{a} > 0,\ \Re e\, s > 0. \tag{2.2}$$

$$\zeta(s) = \frac{3\pi}{a^2} s^{\frac{1}{2}} S_{-2,\frac{1}{2}}(\frac{2s}{a}),\ \Re e\, \mathrm{a} > 0,\ \Re e\, s > 0. \tag{2.3}$$

$$\theta(s) = \frac{8a\pi^{\frac{1}{2}}}{(4s+a^2)^2},\ \Re e\, s > -\Re e\, \frac{a^2}{4}.$$

$$\delta(s) = \frac{32\pi}{3a^3} s^{\frac{3}{2}}\ {}_1F_2[2;\ \frac{7}{4},\ \frac{5}{4};\ -\frac{s^2}{a^2}] + 2(\frac{\pi}{a})^{\frac{3}{2}}\ {}_1F_2[\frac{5}{4};\ \frac{1}{2},\ \frac{1}{4};\ -\frac{s^2}{a^2}] - \frac{6\pi^{\frac{1}{2}}}{a^{\frac{5}{2}}} s\ {}_1F_2[\frac{7}{4};\ \frac{3}{2},\ \frac{3}{4};\ -\frac{s^2}{a^2}],$$

where $\Re e\, \mathrm{a} > 0,\ \Re e\, s > 0.$ (2.4)

Hence,

(a) From (2.1) for $j = 0$ and (2.2), we obtain

$$\mathcal{L}_{\mathrm{n}}\left\{ \frac{\sin[\frac{1}{4}\tan^{-1}\frac{a}{2p_1(\overline{x^{-1}})}]}{p_n(\overline{x^{\frac{1}{2}}})[4p_1^2(\overline{x^{-1}})+a^2]^{\frac{1}{4}}};\ \bar{s} \right\} = (\frac{\pi^{n+1}}{2})^{\frac{1}{2}} \cdot \frac{p_1(\overline{s^{\frac{1}{2}}})}{ap_n(\overline{s^{\frac{1}{2}}})} S_{-1,\frac{1}{2}}[\frac{2}{a}p_1^2(\overline{s^{\frac{1}{2}}})], \tag{2.5}$$

where $\Re e\, a > 0$, $\Re e\, [p_1(\overline{s^{\frac{1}{2}}})] > 0$.

(b) Using (2.1) for $j = 1$ and (2.3), we arrive at

$$\mathcal{L}_n\left\{\frac{\sin[\frac{2}{4}\tan^{-1}\frac{a}{2p_1(\overline{x^{-1}})}]}{p_n(\overline{x^{\frac{1}{2}}})[4p_1^2(\overline{x^{-1}})+a^2]^{\frac{2}{4}}};\ \overline{s}\right\} = \frac{3\pi^{\frac{3}{2}}}{2a^2}\cdot\frac{p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\ S_{-2,\frac{1}{2}}\ [\frac{2}{a}p_1^2(\overline{s^{\frac{1}{2}}})], \tag{2.6}$$

where $\Re e\, a > 0$, $\Re e\, [p_1(\overline{s^{\frac{1}{2}}})] > 0$; and

(c) From (2.1) for $j = 2$ and (2.4), we arrive at

$$\mathcal{L}_n\left\{\frac{\sin[\frac{3}{4}\tan^{-1}\frac{a}{2p_1(\overline{x^{-1}})}]}{p_n(\overline{x^{\frac{1}{2}}})[4p_1^2(\overline{x^{-1}})+a^2]^{\frac{3}{4}}};\ \overline{s}\right\} =$$

$$=\frac{(\frac{2}{a})^{\frac{3}{2}}\pi^{\frac{n-1}{2}}}{p_n(\overline{s^{\frac{1}{2}}})}\{\frac{16}{3a^{\frac{3}{2}}}[p_1(\overline{s^{\frac{1}{2}}})]^3\ {}_1F_2[2;\ \frac{7}{4},\ \frac{5}{4};\ -\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]$$

$$+\pi^{\frac{1}{2}}\ {}_1F_2[\frac{5}{4};\ \frac{1}{2},\ \frac{1}{4};\ -\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]-\frac{3\pi^{\frac{1}{2}}}{a}[p_1(\overline{s^{\frac{1}{2}}})]^2\ {}_1F_2[\frac{7}{4};\ \frac{3}{2},\ \frac{3}{4};\ -\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]\},$$

where $\Re e\, a > 0$, $\Re e\, [p_1(\overline{s^{\frac{1}{2}}})] > 0$. (2.7)

This Theorem can also be used to evaluate N-dimensional Laplace transformations involving generalized hypergeometric functions.

Example 1.3. Consider $f(x) = x^{\tau}\ {}_pF_q\left[\begin{matrix}(a)_p \\ (b)_q\end{matrix};\ kx\right]$. Then

$$\phi(s) = \frac{\Gamma(\tau+1)}{s^{\tau+1}}\ {}_{p+1}F_q\left[\begin{matrix}(a)_p, \tau+1; \frac{k}{s} \\ (b)_q \qquad ;\end{matrix}\right],$$

where $p \le q$, $\Re e\ \tau > -1$ and $\Re e\ s > \begin{cases} 0 \text{ if } p < q \\ \Re e k \text{ if } p = q \end{cases}$.

$$F_j(s)=\frac{\Gamma(\frac{4\tau+j+1}{2})}{s^{\frac{4\tau+j+1}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{4\tau+j+1}{4}, \frac{4\tau+j+3}{4} & ; \frac{4k}{s^2} \\ (b)_q & ;\end{matrix}\right], \tag{3.1}$$

where $p \le q-1$, $\Re e\ \tau > -\frac{j+1}{4}$ for $j = 0, 1, 2$; $\Re e\ s > 0$ if $p \le q$-1; and

$\Re e\ (s + 2k \cos \pi r) > 0$ $(r = 0, 1)$ if $p = q$-1.

$$\xi(s)=\frac{\Gamma(\tau+1)\Gamma(\tau+\frac{1}{2})}{s^{\tau+\frac{1}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2} & ; \frac{k}{s} \\ (b)_q & ;\end{matrix}\right],$$

where $p \le q$-1, $\Re e\ \tau > -\frac{1}{2}$ and $\Re e\ s > \begin{cases} 0 \text{ if } p < q \\ \Re e k \text{ if } p = q \end{cases}$.

$$\eta(s)=\frac{\Gamma(\tau+1)\Gamma(\tau+\frac{1}{2})\Gamma(2\tau+\frac{1}{2})}{s^{2\tau+\frac{1}{2}}}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{1}{4}, \tau+\frac{3}{4} & ; \frac{4k^2}{s^2} \\ (b)_q & ;\end{matrix}\right], \tag{3.2}$$

where $p \le q-3$, $\Re e\ \tau > -\frac{1}{4}$; $\Re e\ s > 0$ if $p \le q-4$; and $\Re e\ (s + 2k \cos \pi r) > 0$ $(r = 0, 1)$ if p $= q$-3.

$$\zeta(s)=\frac{\Gamma(\tau+1)\Gamma(\tau+\frac{1}{2})\Gamma(2\tau+\frac{3}{2})}{s^{2\tau+\frac{3}{2}}}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{3}{4}, \tau+\frac{5}{4} & ; \frac{4k^2}{s^2} \\ (b)_q & ;\end{matrix}\right], \tag{3.3}$$

where $p \le q-3$, $\Re e\ \tau > -\frac{3}{4}$; $\Re e\ s > 0$ if $p \le q-4$; and $\Re e\ (s + 2k \cos \pi r) > 0$ $(r = 0, 1)$ if p $= q$-3.

$$\theta(s)=\frac{\Gamma(\tau+1)\Gamma(\tau+\frac{3}{2})}{s^{\tau+\frac{3}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{3}{2} & ; \frac{k^2}{s} \\ (b)_q & ;\end{matrix}\right],$$

where $p \le q$-1, $\Re e\ \tau > -\frac{3}{2}$ and $\Re e\ s > \begin{cases} 0, \text{ if } p < q\text{-}1 \\ \Re e k, \text{ if } p = q\text{-}1 \end{cases}$.

$$\delta(s)=\frac{\Gamma(\tau+1)\Gamma(\tau+\frac{3}{2})\Gamma(2\tau+\frac{3}{2})}{s^{2\tau+\frac{3}{2}}}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{3}{2}, \tau+\frac{3}{4}, \tau+\frac{5}{4} & ; \frac{4k^2}{s^2} \\ (b)_q & ;\end{matrix}\right], \tag{3.4}$$

where $p \le q-3$, $\Re e\ \tau > -\frac{3}{4}$; $\Re e\ s > 0$ if $p \le q-4$; and $\Re e\ (s + 2k \cos \pi r) > 0$ $(r = 0, 1)$ if p $= q$-3.

Therefore,

(a) In (3.1) set $j = 0$ with (3.2), we have

$$\mathcal{L}_{\mathrm{n}}\left\{\frac{1}{p_n(x^{\frac{1}{2}})[p_1(x^{-1})]^{2\tau+\frac{1}{2}}}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{4\tau+1}{4}, \frac{4\tau+3}{4} & ; \\ (b)_q & ; & \frac{k}{p_1^{\,2}(x^{-1})}\end{matrix}\right]; \bar{s}\right\}$$

$$= \frac{\pi^{\frac{n-1}{2}}\Gamma(2\tau+1)}{p_n(s^{\frac{1}{2}})[p_1(s^{\frac{1}{2}})]^{4\tau+1}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{1}{4}, \tau+\frac{3}{4}, & ; \\ (b)_q & ; & \frac{4k}{p_1^{\,4}(s^{-1})}\end{matrix}\right], \tag{3.5}$$

where $p \le q-3$, $\Re e\ \tau > -\frac{1}{4}$; $\Re e\ [p_1(s^{\frac{1}{2}})] > 0$ if $p \le q-4$; and $\Re e\ [p_1(s^{\frac{1}{2}}) + 2k \cos \pi r] > 0$

$(r = 0, 1)$ if $p = q$-3.

(b) From (3.1) for $j = 1$ and (3.3), we obtain

$$\mathcal{L}_{\mathrm{n}}\left\{\frac{1}{p_n(x^{\frac{1}{2}})[p_1(x^{-1})]^{2\tau+1}}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{4\tau+2}{4}, \frac{4\tau+4}{4} & ; \\ (b)_q & ; & \frac{k}{p_1^{\,2}(x^{-1})}\end{matrix}\right]; \bar{s}\right\}$$

$$= \frac{\pi^{\frac{n-1}{2}}\Gamma(2\tau+\frac{3}{2})}{p_n(s^{\frac{1}{2}})[p_1(s^{\frac{1}{2}})]^{4\tau+2}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{1}{2}, \tau+\frac{3}{4}, \tau+\frac{5}{4}, & ; \\ (b)_q & ; & \frac{4k}{p_1^{\,4}(s^{-1})}\end{matrix}\right], \tag{3.6}$$

where $p \le q-3$, $\Re e\ \tau > -\frac{3}{4}$; $\Re e\ [p_1(s^{\frac{1}{2}})] > 0$ if $p \le q-4$; and $\Re e\ [p_1(s^{\frac{1}{2}}) + 2k \cos \pi r] > 0$

$(r = 0, 1)$ if $p = q$-3.

(c) Using (3.1) for $j = 2$ with (3.4), we arrive at

$$\mathcal{L}_{\mathrm{n}}\left\{\frac{1}{p_n(x^{\frac{1}{2}})[p_1(x^{-1})]^{2\tau+\frac{3}{2}}}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{4\tau+3}{4}, \frac{4\tau+5}{4} & ; \\ (b)_q & ; & \frac{k}{p_1^{\,2}(x^{-1})}\end{matrix}\right]; \bar{s}\right\}$$

$$= \frac{\pi^{\frac{n-1}{2}}\Gamma(2\tau+2)}{\overline{p_n(s^{\frac{1}{2}})}[\overline{p_1(s^{\frac{1}{2}})}]^{4\tau+3}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p, \tau+1, \tau+\frac{3}{2}, \tau+\frac{3}{4}, \tau+\frac{5}{4}, & ; \\ (b)_q & ; & \frac{4k}{\overline{p_1^4(s^{\frac{1}{2}})}}\end{matrix}\right], \tag{3.7}$$

where $p \le q-3$, $\Re e\ \tau > -1$; $\Re e\ [\overline{p_1(s^{\frac{1}{2}})}] > 0$ if $p \le q-4$; and $\Re e\ [\overline{p_1(s^{\frac{1}{2}})} + 2k \cos \pi r] > 0$ $(r = 0, 1)$ if $p = q$-3.

Example 1.4. Suppose that $f(x) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{64} \\ 0 & \text{if } \quad x > \frac{1}{64} \end{cases}$. Then

$$\phi(s) = \frac{1-\exp(-\frac{s}{64})}{s},\ F_j(s) = \frac{\gamma(\frac{j+1}{2}, \frac{s}{8})}{s^{\frac{j+1}{2}}} \text{ for } j = 0,\ 1,\ 2 \text{ and where } \Re e\ s > -\infty. \tag{4.1}$$

Also, we obtain

$$\xi(s) = (\frac{\pi}{s})^{\frac{1}{2}}[1-\exp(-\frac{s^{\frac{1}{2}}}{4})],\ \Re e\ s > 0,$$

$$\eta(s) = \frac{\pi}{s^{\frac{1}{2}}}[1-\exp(-s^{\frac{1}{2}})],\ \Re e\ s > 0, \tag{4.2}$$

$$\zeta(s) = \frac{\pi}{2s^{\frac{3}{2}}}[1-(1+s^{\frac{1}{2}})\exp(-s^{\frac{1}{2}})],\ \Re e\ s > 0, \tag{4.3}$$

$$\theta(s) = \frac{\pi^{\frac{1}{2}}}{2s^{\frac{3}{2}}}[1-(1+\frac{s^{\frac{1}{2}}}{4})\exp(-\frac{s^{\frac{1}{2}}}{4})],\ \Re e\ s > 0,$$

$$\delta(s) = \frac{\pi}{8s^{\frac{3}{2}}}[4-(s^{\frac{1}{2}}+2)^2\exp(-s^{\frac{1}{2}})],\ \Re e\ s > 0. \tag{4.4}$$

Therefore using Theorem 2.1.1, we obtain the following results:

(a) In (4.1) set $j = 0$ with (4.2), we obtain

$$\mathfrak{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{1}{2}}}\ Erf\left[(\frac{p_1(\overline{x^{-1}})}{2})^{\frac{1}{2}}\right];\ \bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-1}{2}}}{p_n(\overline{s^{\frac{1}{2}}})p_1(\overline{s^{\frac{1}{2}}})}\ [1-\exp(-p_1(\overline{s^{\frac{1}{2}}}))], \tag{4.5}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$, $n = 2, 3, \ldots, N$.

(b) Setting $j = 1$ into (4.1) and using (4.3), we obtain

$$\mathfrak{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})p_1(\overline{x^{-1}})}\ [1-\exp(-\frac{p_1(\overline{x^{-1}})}{4})];\ \bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-1}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})p_1(\overline{s^{\frac{1}{2}}})}\ [1-(1+p_1(\overline{s^{\frac{1}{2}}}))\exp(-p_1(\overline{s^{\frac{1}{2}}}))], \tag{4.6}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$, $n = 2, 3, \ldots, N$.

(c) We know

$$\gamma(\frac{3}{2},x)=\frac{1}{2}\gamma(\frac{1}{2},x)-x^{\frac{1}{2}}\exp(-x)=\frac{1}{2}\pi^{\frac{1}{2}}Erf x^{\frac{1}{2}}-x^{\frac{1}{2}}\exp(-x). \tag{4.7}$$

Setting $j = 2$ into (4.1) and using (4.7) and (4.3) we arrive at

$$\mathfrak{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})}\ \left\{\frac{\pi^{\frac{1}{2}}}{2}Erf[(\frac{p_1(\overline{x^{-1}})}{4})^{\frac{1}{2}}]-[\frac{p_1(\overline{x^{-1}})}{4}]^{\frac{1}{2}}\exp(-\frac{p_1(\overline{x^{-1}})}{4})\right\};\ \bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-1}{2}}}{2^{\frac{1}{2}}\cdot 8p_n(\overline{s^{\frac{1}{2}}})p_1^{\,3}(\overline{s^{\frac{1}{2}}})}\ [4-(2+p_1(\overline{s^{\frac{1}{2}}}))^2\exp(-p_1(\overline{s^{\frac{1}{2}}}))], \tag{4.8}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$, $n = 2, 3, \ldots, N$.

2.1.1.2. Corollaries

On choosing $n = 2$, we obtain certain corollaries from Theorem 2.1.1, namely:

Corollary 1. Assume the hypotheses of Theorem 2.1.1(a) except for condition (*ii*), which is replaced by

(*ii*)(a) $$\mathfrak{L}_2\left\{x^{-\frac{1}{2}}f(x^2);s\right\} = F_0(s).$$

Then

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}}F_0[\frac{2}{x}+\frac{2}{y}];s_1,s_2\right\} = \frac{1}{2^{\frac{1}{2}}(s_1s_2)^{\frac{1}{2}}}\eta[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2],\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$$

Corollary 2. Suppose that the hypotheses of Theorem 2.1.1 (b) except for condition (*ii*) which is replaced by

(*ii*)(b) $$\mathfrak{L}_2\left\{f(x^2);s\right\} = F_1(s).$$

Then

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}}F[\frac{2}{x}+\frac{2}{y}];s_1,s_2\right\} = \frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}{2(s_1s_2)^{\frac{1}{2}}}\zeta[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2],\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$$

Corollary 3. Assume the hypotheses (*i*), (*iii*) and (*iv*) of Theorem 2.1.1 (c) and replace condition (*ii*) by

(*ii*)(c) $$\mathfrak{L}_2\left\{x^{\frac{1}{2}}f(x^2);s\right\} = F_2(s).$$

Then

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}}F_2[\frac{2}{x}+\frac{2}{y}];s_1,s_2\right\} = \frac{1}{2^{\frac{1}{2}}(s_1s_2)^{\frac{1}{2}}}\gamma[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2],\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$$

2.1.1.3. Examples Based Upon Corollaries 1, 2, and 3

Example 1.1.1. Let $n = 2$, from Example 1.1, the following results are yielded

(a')

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}(\frac{1}{x}+\frac{1}{y})^{\frac{\upsilon+1}{2}}},s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{\upsilon}{2}+1)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\upsilon+1}} \qquad (1.1')$$

or

$$\mathfrak{L}_2\left\{\frac{(xy)^{\frac{v}{2}}}{(x+y)^{\frac{v+1}{2}}};s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{v}{2}+1)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{v+1}},$$

where $\Re e\, v>-1$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

(b')

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}(\frac{1}{x}+\frac{1}{y})^{\frac{v+2}{2}}},s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{v}{2}+\frac{3}{2})\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{v+2}}$$

or (1.2')

$$\mathfrak{L}_2\left\{\frac{(xy)^{\frac{v+1}{2}}}{(x+y)^{\frac{v+2}{2}}};s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{v}{2}+\frac{3}{2})\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{v+2}},$$

where $\Re e\, v>-2$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

(c')

$$\mathfrak{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}(\frac{1}{x}+\frac{1}{y})^{\frac{v+3}{2}}},s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{v}{2}+2)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{v+3}}$$

or (1.3')

$$\mathfrak{L}_2\left\{\frac{(xy)^{\frac{v+2}{2}}}{(x+y)^{\frac{v+3}{2}}};s_1,s_2\right\} = \pi^{\frac{1}{2}}\Gamma(\frac{v}{2}+2)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{v+3}},$$

where $\Re e\, v>-3$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Remark 2.1.1.2: If we let $v = 0$ in (1.1'), we obtain

$$\mathfrak{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}},s_1,s_2\right\} = \frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}. \tag{1.1''}$$

From (1.1'') and the following relations

$$\mathfrak{L}_2\left\{xF(x,y);s_1,s_2\right\} = -\frac{\partial}{\partial s_1}f(s_1,s_2),$$

$$\mathfrak{L}_2\left\{yF(x,y);s_1,s_2\right\} = \frac{\partial}{\partial s_2}f(s_1,s_2), \text{ where } f(s_1,s_2)=\mathfrak{L}_2\left\{F(x,y);s_1,s_2\right\}$$

We conclude that

$$\mathcal{L}_2\left\{(x+y)^{\frac{1}{2}};s_1,s_2\right\} = \frac{\pi^{\frac{1}{2}}(s_1+s_2+s_1^{\frac{1}{2}}s_2^{\frac{1}{2}})}{2(s_1s_2)^{\frac{3}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})} \tag{1.1'''}$$

Example 1.2.1. In Example 1.2, if $n = 2$, the following results can be obtained

(a')

$$\mathcal{L}_2\left\{\frac{\sin[\frac{1}{4}\tan^{-1}\frac{axy}{2(x+y)}]}{[4(x+y)^2+a^2(xy)^2]^{\frac{1}{4}}};s_1,s_2\right\} = (\frac{\pi^3}{2})^{\frac{1}{2}}\cdot\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{a(s_1s_2)^{\frac{1}{2}}}\, S_{-1,\frac{1}{2}}\,[\frac{2}{a}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2],$$

where $\Re e\, a>0$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$. (2.5')

(b')

$$\mathcal{L}_2\left\{\frac{\sin[\frac{2}{4}\tan^{-1}\frac{axy}{2(x+y)}]}{[4(x+y)^2+a^2(xy)^2]^{\frac{1}{4}}};s_1,s_2\right\} = \frac{3\pi}{2a^2}\cdot\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}}\, S_{-2,\frac{1}{2}}\,[\frac{2}{a}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2],$$

where $\Re e\, a>0$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$. (2.6')

(c')

$$\mathcal{L}_2\left\{\frac{\sin[\frac{3}{4}\tan^{-1}\frac{axy}{2(x+y)}]}{[4(x+y)^2+a^2(xy)^2]^{\frac{1}{4}}};s_1,s_2\right\} = \frac{(\frac{2}{a})^{\frac{3}{2}}\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}}\,\{\frac{16}{3a^{\frac{3}{2}}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3\,{}_1F_2[2;\frac{7}{4},\frac{5}{4};-\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}{a^2}]$$

$$+\pi^{\frac{1}{2}}\,{}_1F_2[\frac{5}{4};\frac{1}{2},\frac{1}{4};-\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}{a^2}]-\frac{3\pi^{\frac{1}{2}}}{a}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\,{}_1F_2[\frac{7}{4};\frac{3}{2},\frac{3}{4};-\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}{a^2}]\}, \tag{2.7'}$$

where $\Re e\, a>0$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Example 1.3.1. On choosing $n = 2$, we obtain the following formulas from Example 1.3, namely:

(a')

$$\mathcal{L}_2\left\{\frac{(xy)^{2\tau}}{(x+y)^{2\tau+\frac{1}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{4\tau+1}{4},\frac{4\tau+3}{4}; \\ (b)_q\qquad\qquad ;\end{matrix} k(\tfrac{xy}{x+y})^2\right];s_1,s_2\right\}$$

$$= \pi^{\frac{1}{2}}\Gamma(2\tau+1)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{4\tau+1}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\tau+1,\tau+\frac{1}{2},\tau+\frac{1}{4},\tau+\frac{3}{4};\\(b)_q\end{matrix}\ \begin{matrix}\\;\frac{4k}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}\end{matrix}\right],$$

where $p \le q-3$, $\Re e\ \tau > -\frac{1}{4}$; $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > 0$ if $p \le q-4$; and $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+2k\cos \pi r] > 0$

$(r = 0,\ 1)$ if $p = q-3$. (3.5')

(b')

$$\mathfrak{L}_2\left\{\frac{(xy)^{2\tau+\frac{1}{2}}}{(x+y)^{2\tau+1}}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{4\tau+2}{4},\frac{4\tau+4}{4};\\(b)_q\ ;\end{matrix}\ k(\frac{xy}{x+y})^2\right];s_1,s_2\right\}$$

$$= \pi^{\frac{1}{2}}\Gamma(2\tau+\frac{3}{2})\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{4\tau+2}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\tau+1,\tau+\frac{1}{2},\tau+\frac{3}{4},\tau+\frac{5}{4};\\(b)_q\end{matrix}\ \begin{matrix}\\;\frac{4k}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}\end{matrix}\right],$$

where $p \le q-3$, $\Re e\ \tau > -\frac{3}{4}$; $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > 0$ if $p \le q-4$; and $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+2k\cos \pi r] > 0$

$(r = 0,\ 1)$ if $p = q-3$. (3.6')

(c')

$$\mathfrak{L}_2\left\{\frac{(xy)^{2\tau+1}}{(x+y)^{2\tau+\frac{3}{2}}}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{4\tau+3}{4},\frac{4\tau+5}{4};\\(b)_q\ ;\end{matrix}\ k(\frac{xy}{x+y})^2\right];s_1,s_2\right\}$$

$$= \pi^{\frac{1}{2}}\Gamma(2\tau+2)\cdot\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{4\tau+3}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\tau+1,\tau+\frac{3}{2},\tau+\frac{3}{4},\tau+\frac{5}{4};\\(b)_q\end{matrix}\ \begin{matrix}\\;\frac{4k}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}\end{matrix}\right],$$

where $p \le q-3$, $\Re e\ \tau > -1$; $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > 0$ if $p \le q-4$; and $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+2k\cos \pi r] > 0$

$(r = 0,\ 1)$ if $p = q-3$. (3.7')

Example 1.4.1. Let us choose $n = 2$ in Example 1.4, we arrive at the following results in two-dimensions.

(a')

$$\mathfrak{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}}\ \mathrm{Erf}\left[(\frac{x+y}{2xy})^{\frac{1}{2}}\right];s_1,s_2\right\}$$

$$= \frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}\left[1-\exp[-(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]\right], \text{ where } \Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > 0. \qquad (4.5')$$

(b')

$$\mathcal{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}}[1-\exp(-\frac{x+y}{4xy})];s_1,s_2\right\}$$

$$=\frac{\pi^{\frac{1}{2}}}{2(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}[1-(1+s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})\exp[-(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]], \text{ where } \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (4.6')$$

(b") Because

$$\mathcal{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}};s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}$$

then (4.6′) also can be written as

$$\mathcal{L}_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}}\exp(-\frac{x+y}{4xy});s_1,s_2\right\}$$

$$=\frac{\pi^{\frac{1}{2}}}{2(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{4\tau+3}}\left[1+(1+s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})\exp[-(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]\right],$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$ (4.6")

(c') If we choose $n = 2$ in (4.8), we obtain

$$\mathcal{L}_2\left\{\frac{1}{(xy)^{\frac{1}{2}}}\left\{\frac{\pi^{\frac{1}{2}}}{2}Erf[(\frac{x+y}{4xy})^{\frac{1}{2}}]-[\frac{x+y}{4xy}]^{\frac{1}{2}}\exp(-\frac{x+y}{4xy})\right\};\bar{s}\right\}$$

$$=(\frac{\pi}{2})^{\frac{1}{8}}\frac{1}{8(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3}[4-(2+s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\exp[-(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]],$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$ (4.8')

Remark 2.1.1.3: The results that are established in Examples 1.2.1, 1.3.1 and 1.4.1 are all new formulas in two-dimensions as well as the corresponding results in n-dimensions.

2.1.2. The Original of Functions with the Argument $[p_1(\overline{s^{\frac{1}{2}}})]^2$

The goals of this part of Section One are to obtain the inverse Laplace transformations of functions of the form $\dfrac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2]$, $\dfrac{[p_1(\overline{s^{\frac{1}{2}}})]^5}{p_n(\overline{s^{\frac{1}{2}}})}\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2]$, and $\dfrac{\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]}{p_n(\overline{s^{\frac{1}{2}}})}$ and establish several new formulas for calculating inverse Laplace transform pairs of N-dimensions from known one-dimensional Laplace transform. Regarding the application of these formulas, we will consider several examples demonstrating ways of obtaining new N-dimensional transform pairs.

Theorem 2.1.2.

Suppose that (*i*) $\mathcal{L}\{f(x); s\} = \phi(s)$ and that

(*ii*) $\mathcal{L}\{x^k f(x^2); s\} = H_k(s)$ for $k = 1,2,3$.

Let $f(x^2)$, $x^{-\frac{1}{2}}\phi(\frac{1}{x})$ and $x^{-\frac{1}{2}}\xi(\frac{1}{x^2})$ belong to the class Ω.

(a) If

(a1) $\mathcal{L}\left\{x^{-\frac{1}{2}}\phi(\frac{1}{x}); s\right\} = \xi(s)$ and

(a2) $\mathcal{L}\left\{x^{-\frac{1}{2}}\xi(\frac{1}{x^2}); s\right\} = \eta(s)$,

Let $x^{-\frac{1}{2}}\exp(-sx-\frac{u}{x})f(u)$ and $x^{\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$.

Then

$$\mathcal{L}_n^{-1}\left\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2]; \overline{x}\right\}$$

$$= \frac{1}{\pi^{\frac{n-2}{2}} p_n(\overline{x^{\frac{1}{2}}})} \{H_1[2p_1(\overline{x^{-1}})] + 4p_1(\overline{x^{-1}}) H_2[2p_1(\overline{x^{-1}})]\}. \tag{1.2.1}$$

It is assumed that the integrals involved exist. The existence conditions for two-dimensions are given in Ditkin and Prudnikov [43; p. 32]. Similar conditions are also true for N-dimensions . For details we refer to Brychkov et al. [11; ch.2].

(b) Assume that the conditions (i), (ii), and (a1) hold. If

(b1) $\mathcal{L}\left\{ x^{\frac{1}{2}} \xi(\frac{1}{x^2}); s \right\} = \zeta(s),$

suppose that $x^{\frac{3}{2}} \exp(-sx - \frac{2u^{\frac{1}{2}}}{x}) f(u)$ belongs to $L_1(0,\infty)\times(0,\infty)]$.

Then

$$\mathcal{L}_n^{-1}\left\{ \frac{[p_1(\overline{s^{\frac{1}{2}}})]^5}{p_n(\overline{s^{\frac{1}{2}}})} \zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2]; \bar{x} \right\}$$

$$= \frac{1}{\pi^{\frac{n-2}{2}} p_n(\overline{x^{\frac{1}{2}}})} \{\frac{3}{2} H_1[2p_1(\overline{x^{-1}})] + 4p_1(\overline{x^{-1}}) H_2[2p_1(\overline{x^{-1}})]$$

$$+ 8[p_1(\overline{x^{-1}})]^2 H_3[2p_1(\overline{x^{-1}})]\}, \tag{1.2.2}$$

provided the integrals involved exist.

Proof (a): First, we apply the definition of Laplace transform to (i) to obtain $\phi(s) = \int_0^\infty \exp(-st) f(t) dt$, $\Re e\, s > c_0$, such that

$$\int_0^\infty x^{-\frac{1}{2}} \phi(\frac{1}{x}) \exp(-sx) dx = \int_0^\infty [\int_0^\infty x^{-\frac{1}{2}} \exp(-sx - \frac{u}{x}) f(u) du] dx. \tag{1.2.3}$$

The integerand $x^{-\frac{1}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs $L_1[(0,\infty)\times(0,\infty)]$, and thus applying Fubini's Theorem on the right hand side of (1.2.3) and using (a1) on the left side of (1.2.3) yields

$$\xi(s)=\int_0^\infty f(u)[\int_0^\infty x^{-\frac{1}{2}}\exp(-sx-\frac{u}{x})dx]du,\ \Re e\ s>c_1, \text{ where } c_1 \text{ is a constant.} \tag{1.2.4}$$

From the tables by Roberts and Kaufman [87], we obtain

$$\xi(s)=\pi^{\frac{1}{2}}s^{-\frac{1}{2}}\int_0^\infty \exp(-2u^{\frac{1}{2}}s^{\frac{1}{2}})f(u)du,\ \Re e\ s>c_1. \tag{1.2.5}$$

Using (1.2.5) in (a2), we arrive at

$$\eta(s)=\pi^{\frac{1}{2}}\int_0^\infty[\int_0^\infty x^{\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)du]dx,\ \Re e\ s>c_2, \text{ where } c_2 \text{ is a constant.} \tag{1.2.6}$$

Because $x^{\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, we can apply Fubini's Theorem on the right of (1.2.6) to interchange the order of integration to obtain

$$\eta(s)=\pi^{\frac{1}{2}}\int_0^\infty f(u)[\int_0^\infty x^{\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})dx]du. \tag{1.2.7}$$

Again, we use Roberts and Kaufman [87] results on the inner integral in (1.2.7), which brings us to

$$s^{\frac{3}{2}}\eta(s)=\frac{\pi}{2}\int_0^\infty f(u)[1+2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}}]\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du. \tag{1.2.8}$$

By substituting $u=\upsilon^2$ into (1.2.8) and then replacing s with $[\overline{p_1(s^{\frac{1}{2}})}]^2$ and multiplying by $p_n(\overline{s^{\frac{1}{2}}})$, we obtain

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^3\,\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2]=\pi\int_0^\infty vf(v^2)[1+2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})]\cdot p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))dv, \quad (1.2.9)$$

Using following two operational results given in Ditkin and Prudnikov [43]

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}}) \doteqdot (\pi x_i)^{-\frac{1}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, \text{n and}$$

$$s_i\exp(-as_i^{\frac{1}{2}}) \doteqdot \frac{a}{2x^{\frac{1}{2}}}x_i^{-\frac{3}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, \text{n},$$

equation (1.2.9) can be rewritten as

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^3\,\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \underset{n}{\overset{n}{=}} \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{\int_0^\infty vf(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv,$$

$$+4p_1(\overline{x^{-1}})\int_0^\infty v^2f(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv\}, \quad (1.2.10)$$

From (*ii*) for $k = 1, 2$, equation (1.2.10) reads

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^3\,\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2]$$

$$\underset{n}{\overset{n}{=}} \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{H_1[2p_1(\overline{x^{-1}})]+4p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})];$$

therefore,

$$\mathcal{L}_n^{-1}\left\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^2];\bar{x}\right\}$$

$$=\frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{H_1[2p_1(\overline{x^{-1}})]+4p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})]\},$$

where $n = 2, 3, \ldots, N$.

Proof (b): From (1.2.5) and (b1), we find

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty[\int_0^\infty x^{\frac{3}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)du]dx, \tag{1.2.11}$$

where $\Re e\ s > c_2$. By the assumption $x^{\frac{3}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, apply Fubini's Theorem, the order of the integral on the right side of (1.2.11) can be interchanged to obtain

$$\zeta(s)=\pi^{\frac{1}{2}}\int_0^\infty f(u)[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})dx]du. \tag{1.2.12}$$

By using the result from Roberts and Kaufman [87], we can evaluate the inner integral in (1.2.12) and obtain

$$s^{\frac{5}{2}}\zeta(s)=\frac{\pi}{4}[\int_0^\infty[3+6\cdot 2^{\frac{1}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}}+8u^{\frac{1}{2}}s]f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du]. \tag{1.2.13}$$

Now, we substitute $u=v^2$ into (1.2.13) and then we replace s with $[p_1(\overline{s^{\frac{1}{2}}})]^2$ and multiplying both sides of (1.2.13) by $p_n(\overline{s^{\frac{1}{2}}})$ to find

$$\begin{aligned}p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^5\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2]&=\frac{\pi}{2}\{3\int_0^\infty vf(v^2)p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))dv\\&+6\cdot 2^{\frac{1}{2}}\int_0^\infty v^{\frac{3}{2}}f(v^2)p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))dv\\&+8\int_0^\infty v^2f(v^2)[p_1(\overline{s^{\frac{1}{2}}})]^2p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))dv\}.\end{aligned} \tag{1.2.14}$$

If we let $-2^{\frac{3}{2}}v^{\frac{1}{2}}s_i^{\frac{1}{2}}=A_i$ for $i = 1, 2, \ldots, n$ then (1.2.14) can be written

$$\begin{aligned}&p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^5\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2]=\frac{\pi}{2}\{3\int_0^\infty vf(v^2)\cdot s_1^{\frac{1}{2}}\exp(A_1)\cdot s_2^{\frac{1}{2}}\exp(A_2)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)dv\\&+6\cdot 2^{\frac{1}{2}}\int_0^\infty v^{\frac{3}{2}}f(v^2)\{[s_1\exp(A_1)\cdot s_2^{\frac{1}{2}}\exp(A_2)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)]+\ldots\\&+[s_n\exp(A_n)\cdot s_1^{\frac{1}{2}}\exp(A_1)\ \ldots\ s_{n-1}^{\frac{1}{2}}\exp(A_{n-1})]\}dv+8[\int_0^\infty v^2f(v^2)\\&\{[s_1^{\frac{3}{2}}\exp(A_1)s_2^{\frac{1}{2}}\exp(A_2)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)]+\ \ldots\ +[s_n^{\frac{3}{2}}\exp(A_n)\cdot s_1^{\frac{1}{2}}\exp(A_1)\\&\ldots\ s_{n-1}^{\frac{1}{2}}\exp(A_{n-1})]\}dv+2\int_0^\infty v^2f(v^2)\{[s_1\exp(A_1)\cdot s_2\exp(A_2)\end{aligned}$$

$$\cdot s_3^{\frac{1}{2}}\exp(A_3)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)]+[s_1\exp(A_1)\cdot s_3\exp(A_3)\cdot s_2^{\frac{1}{2}}\exp(A_2)$$

$$\ldots\ s_n^{\frac{1}{2}}\exp(A_n)]+\ \ldots\ +[s_1\exp(A_1)\cdot s_n\exp(A_n)\cdot s_2^{\frac{1}{2}}\exp(A_2)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)]\}$$

$$+\{[s_2\exp(A_2)\cdot s_3\exp(A_3)\cdot s_1^{\frac{1}{2}}\exp(A_1)\ldots\ s_n^{\frac{1}{2}}\exp(A_n)]$$

$$+[s_2\exp(A_2)\cdot s_4\exp(A_4)\cdot s_1^{\frac{1}{2}}\exp(A_1)\ \ldots\ s_n^{\frac{1}{2}}\exp(A_n)]+\ \ldots$$

$$+[s_2\exp(A_2)\cdot s_n\exp(A_n)\cdot s_1^{\frac{1}{2}}\exp(A_1)\ \ldots\ s_{n-1}^{\frac{1}{2}}\exp(A_{n-1})]+\ \ldots$$

$$+[s_{n-1}\exp(A_{n-1})\cdot s_n\exp(A_n)\cdot s_1^{\frac{1}{2}}\exp(A_1)\ \ldots\ s_{n-2}^{\frac{1}{2}}\exp(A_{n-2})]\}\}dv. \qquad (1.2.14')$$

Now we use operational results given in Ditkin and Prudnikov [43]:

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}}) \doteq (\pi x_i)^{-\frac{1}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, n,$$

$$s_i\exp(-as_i^{\frac{1}{2}}) \doteq \frac{a}{2\pi^{\frac{1}{2}}}x_i^{-\frac{3}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, n,$$

$$s_i^{\frac{3}{2}}\exp(-as_i^{\frac{1}{2}}) \doteq \frac{a^2-2x_i}{4\pi^{\frac{1}{2}}}x_i^{-\frac{5}{2}}\exp(-\frac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, n.$$

Equation (1.2.14') can be evaluated as

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^5\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \underset{n}{\overset{n}{=}} \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{\frac{3}{2}\int_0^\infty vf(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv$$

$$+\ 4p_1(\overline{x^{-1}})\int_0^\infty v^2f(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv$$

$$+\ 8[p_1(\overline{x^{-1}})]^2\int_0^\infty v^3f(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv\}. \qquad (1.2.15)$$

Using (*ii*) for $k = 1, 2, 3$, equation (1.2.15) can be written as

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^5\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2] \underset{n}{\overset{n}{=}} \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{\frac{3}{2}H_1[2p_1(\overline{x^{-1}})]$$

$$+\ 4p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})]+\ 8[p_1(\overline{x^{-1}})]^2H_3[2p_1(\overline{x^{-1}})]\}.$$

Hence,

$$\mathcal{L}_n^{-1}\left\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^5}{p_n(\overline{s^{\frac{1}{2}}})}\zeta[(p_1(\overline{s^{\frac{1}{2}}}))^2];\bar{x}\right\}$$

$$=\frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\left\{\frac{3}{2}H_1[2p_1(\overline{x^{-1}})]+4p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})]\right.$$

$$\left.+8[p_1(\overline{x^{-1}})]^2H_3[2p_1(\overline{x^{-1}})]\right.$$

2.1.2.1. Applications of Theorem 2.1.2

Next we will provide examples of the applications of Theorem 2.1.2.

Example 2.1. Suppose that $f(x)=\cosh(ax^{\frac{1}{2}})$; then

$$\phi(s)=\frac{\pi^{\frac{1}{2}}a}{2s^{\frac{3}{2}}}\exp(\frac{a^2}{4s})Erf(\frac{a}{2s^{\frac{1}{2}}})+\frac{1}{s},\ \Re e\, s>0.$$

$$H_k(s)=\frac{\Gamma(k+1)}{2}[(s-a)^{-k-1}+(s+a)^{-k-1}]\text{ for } k=1,\ 2,\ 3, \text{ and } \Re e\, s>|\Re e a|, \text{ and}$$

$$\xi(s)=\pi^{\frac{1}{2}}[\frac{1}{s^{\frac{3}{2}}}-\frac{1}{2s^{\frac{1}{2}}(\frac{a}{2}+s^{\frac{1}{2}})^2}],\ \Re e\, s>\begin{cases}0 & \text{if } \Re e\, a\geq 0\\ \max\{0,\ \Re e\frac{a^2}{4}\} & \text{if } \Re e\, a<0\end{cases}.$$

$$\eta(s)=\frac{15\pi}{8}2^{-\frac{7}{2}}[1-\frac{1}{2}(\frac{2}{a})^{\frac{9}{4}}s^{\frac{9}{4}}\exp(\frac{s}{a})W_{-\frac{9}{4},\frac{3}{4}}(\frac{2s}{a})],|\arg a|<\pi,\ \Re e\, s>\begin{cases}0 & \text{if } \Re e\, a\geq 0\\ \max\{0,\ \Re e\frac{a^2}{4}\} & \text{if } \Re e\, a<0\end{cases}.$$

$$\zeta(s)=\frac{105\pi}{16}s^{-\frac{9}{2}}[1-\frac{1}{2}(\frac{2}{a})^{\frac{11}{4}}s^{\frac{11}{4}}\exp(\frac{s}{a})W_{-\frac{11}{4},\frac{5}{4}}(\frac{2s}{a})],|\arg a|<\pi,\ \Re e\, s>\begin{cases}0 & \text{if } \Re e\, a\geq 0\\ \max\{0,\ \Re e\frac{a^2}{4}\} & \text{if } \Re e\, a<0\end{cases}.$$

Hence,

(a) $$\mathcal{L}_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^4p_n(\overline{s^{\frac{1}{2}}})}[1-\frac{1}{2}(\frac{2}{a})^{\frac{9}{4}}[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{9}{2}}\exp[\frac{(p_1(\overline{s^{\frac{1}{2}}}))^2}{a}]\cdot W_{-\frac{9}{4},\frac{3}{4}}[\frac{2(p_1(\overline{s^{\frac{1}{2}}}))^2}{a}]];\bar{x}\right\}$$

$$=\frac{12}{5\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{[2p_1(\overline{x^{-1}})-a]^{-2}+[2p_1(\overline{x^{-1}})+a]^{-2}\}+4p_1(\overline{x^{-1}})\{[2p_1(\overline{x^{-1}})-a]^{-3}+[2p_1(\overline{x^{-1}})+a]^{-3}\},$$

where $|\arg a|<\pi,\ \Re e\,[p_1(\overline{s^{\frac{1}{2}}})]>\begin{cases}0 & \text{if } \Re e\, a\geq 0\\ \max\{0,\ \Re e\frac{a^2}{4}\} & \text{if } \Re e\, a<0\end{cases},\ n\ =\ 2,\ 3,\ K\ ,\ N.$ (1.1)

$$\text{(b)}\quad \mathcal{L}_{\mathrm{n}}^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^4\, p_n(\overline{s^{\frac{1}{2}}})}[1-\frac{1}{2}(\frac{2}{a})^{\frac{11}{4}}[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{11}{4}}\exp[\frac{(p_1(\overline{s^{\frac{1}{2}}}))^2}{a}]\cdot W_{-\frac{11}{4},\frac{5}{4}}[\frac{2(p_1(\overline{s^{\frac{1}{2}}}))^2}{a}]];\overline{x}\right\}$$

$$=\frac{16}{105\pi^{\frac{4}{2}}\, p_n(\overline{x^{\frac{1}{2}}})}\{\frac{3}{4}\{[2p_1(\overline{x^{-1}})-a]^{-2}+[2p_1(\overline{x^{-1}})+a]^{-2}\}+4p_1(\overline{x^{-1}})\{[2p_1(\overline{x^{-1}})-a]^{-3}+[2p_1(\overline{x^{-1}})+a]^{-3}\},$$

$$+\ 24\,[p_1(\overline{x^{-1}})]^2\cdot\{[2p_1(\overline{x^{-1}})-a]^{-4}+[2p_1(\overline{x^{-1}})+a]^{-4}\}\},$$

where $|\arg a|<\pi$, $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > \begin{cases} 0 & \text{if } \Re e\, a \ge 0 \\ \max\{0, \Re e\frac{a^2}{4}\} & \text{if } \Re e\, a<0 \end{cases}$ (1.2)

Example 2.2. Replacing f with the following

(*i*) $f(x)=J_0(ax^{\frac{1}{2}})$ or

(*ii*) $f(x)=x^{\alpha}\ {}_pF_q[{(a)_p\,;\atop (b)_q\,;}\ lx]$

in Theorem 2.1.2, we arrive at the following N-dimensional inverse Laplace transformation pairs:

$$\text{(a)}(i)\quad \mathcal{L}_{\mathrm{n}}^{-1}\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\{[p_1(\overline{s^{\frac{1}{2}}})]\ {}_1F_2[\frac{6}{4};\frac{3}{4},\frac{1}{4};-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]+\frac{4}{a^3}(\frac{\pi}{2a})^{\frac{1}{2}}\beta(-\frac{1}{4},\frac{7}{4})\ {}_1F_1[\frac{7}{4};\frac{2}{5},\frac{5}{4};$$

$$-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]-\frac{\pi^{\frac{1}{2}}}{a^2}(\frac{2}{a})^{\frac{5}{2}}\beta(-\frac{3}{4},\frac{9}{4})[p_1(\overline{s^{\frac{1}{2}}})]^2\ {}_1F_2[\frac{9}{4};\frac{6}{4},\frac{7}{4};-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^2}{a^2}]\};\overline{x}\}$$

$$=\frac{1}{\pi^{\frac{n-2}{2}}\, p_n(\overline{x^{\frac{1}{2}}})}\{[(2p_1(\overline{x^{-1}}))^2+a^2]^{-1}\ {}_2F_1[1,-\frac{1}{2};1;\frac{a^2}{[2p_1(\overline{x^{-1}})]^2+a^2}$$

$$+32p_1(\overline{x^{-1}})[(2p_1(\overline{x^{-1}}))^2+a^2]^{-\frac{3}{2}}\ {}_2F_1[\frac{3}{2},-1;1;\frac{a^2}{[2p_1(\overline{x^{-1}})]^2+a^2}]\},$$

where $\Re e\, a>0$, $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})]>|\mathrm{Im}\ a|$. (2.1)

$$\text{(b)}(i)\quad \mathcal{L}_{\mathrm{n}}^{-1}\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^5}{p_n(\overline{s^{\frac{1}{2}}})}\{\frac{2\pi}{a^3}[p_1(\overline{s^{\frac{1}{2}}})]^{-\frac{3}{2}}\ {}_1F_2[\frac{6}{4};-\frac{1}{4},-\frac{1}{4};-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]$$

$$+\frac{\pi^{\frac{1}{2}}}{a}(\frac{2}{a})^{\frac{7}{2}}\beta(-\frac{3}{4},\frac{9}{4})\ {}_1F_2[\frac{9}{4};\frac{2}{4},\frac{13}{4};-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}]$$

$$-\left(\frac{\pi}{a}\right)^{\frac{1}{2}}\left(\frac{2}{a}\right)^{\frac{7}{2}}\beta\left(-\frac{5}{4},\frac{11}{4}\right)p_1(\overline{s^{\frac{1}{2}}})\ {}_1F_2\left[\frac{11}{4};\frac{6}{4},\frac{9}{4};-\frac{[p_1(\overline{s^{\frac{1}{2}}})]^4}{a^2}\right];\overline{x}\Big\}$$

$$=\frac{1}{\pi^{\frac{n-2}{2}}\,\overline{p_n(x^{\frac{1}{2}})}}\Big\{\frac{3}{2}\left[(2p_1(\overline{x^{-1}}))^2+a^2\right]^{-1}\ {}_2F_1\left[1,-\frac{1}{2};1;\frac{a^2}{[2p_1(\overline{x^{-1}})]^2+a^2}\right]$$

$$+\ 8p_1(\overline{x^{-1}})\left[(2p_1(\overline{x^{-1}}))^2+a^2\right]^{-\frac{3}{2}}\ {}_2F_1\left[\frac{3}{2},-1;1;\frac{a^2}{[2p_1(\overline{x^{-1}})]^2+a^2}\right]$$

$$+\ 48[p_1(\overline{x^{-1}})]^2\left[(2p_1(\overline{x^{-1}}))^2+a^2\right]^{-2}\ {}_2F_1\left[2,-1;1;\frac{a^2}{[2p_1(\overline{x^{-1}})]^2+a^2}\right]\Big\},$$

where $\Re e\ a>0$, $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})]>|\mathrm{Im}\ a|$. (2.2)

(a)(*ii*)

$$\mathfrak{L}_n^{-1}\Big\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{4\alpha+4}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{4}{4},\alpha+\frac{6}{4},\alpha+\frac{7}{4},\alpha+\frac{9}{4};\\(b)_q\end{matrix}\ ;\frac{4l}{[p_1(\overline{s^{\frac{1}{2}}})]^4}\right];\overline{x}\Big\}$$

$$=\frac{1}{2\pi^{\frac{n-1}{2}}\Gamma(2\alpha+\frac{7}{2})p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{2\alpha+2}}\Big\{{}_{p+2}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{2}{2},\alpha+\frac{3}{2};\\(b)_q\end{matrix}\ ;\frac{l}{[p_1(\overline{x^{-1}})]^2}\right]$$

$$+8(\alpha+1)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{3}{2},\alpha+\frac{4}{2};\\(b)_q\end{matrix}\ ;\frac{l}{[p_1(\overline{x^{-1}})]^2}\right]\Big\},$$

where $p\le q-3$, $\Re e\ \alpha>-1$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0$ if $p\le q-4$, and $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})+2l\cos\pi r]>0$

$(r=0,\ 1)$ if $p=q-3$. (2.3)

(b)(*ii*)

$$\mathfrak{L}_n^{-1}\Big\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{4\alpha+4}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{4}{4},\alpha+\frac{6}{4},\alpha+\frac{9}{4},\alpha+\frac{11}{4};\\(b)_q\end{matrix}\ ;\frac{4l}{[p_1(\overline{s^{\frac{1}{2}}})]^4}\right];\overline{x}\Big\}$$

$$=\frac{1}{2\pi^{\frac{n-1}{2}}\Gamma(2\alpha+\frac{9}{2})p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{2\alpha+2}}\Big\{\frac{3}{2}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{2}{2},\alpha+\frac{3}{2};\\(b)_q\end{matrix}\ ;\frac{l}{[p_1(\overline{x^{-1}})]^2}\right]$$

$$+4(\alpha+1)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{3}{2},\alpha+\frac{4}{2};\\(b)_q\end{matrix}\ ;\frac{l}{[p_1(\overline{x^{-1}})]^2}\right]$$

$$+4(\alpha+1)(2\alpha+3)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\alpha+\frac{4}{2},\alpha+\frac{5}{2};\\(b)_q\end{matrix}\ ;\frac{l}{[p_1(\overline{x^{-1}})]^2}\right]\Big\},$$

where $p\le q-3$, $\Re e\ \alpha>-1$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0$ if $p\le q-4$, and $\Re e\ [p_1(\overline{s^{\frac{1}{2}}})+2l\cos\pi r]>0$

$(r=0,\ 1)$ if $p=q-3$. (2.4)

2.1.2.2. Corollaries

A number of corollaries can be derived from Theorem 2.1.2 in two dimensions:

Corollary 1. Suppose that the assumptions of Theorem 2.1.2(a) hold except for condition (ii) which holds for k = 1, 2. Then

$$\mathcal{L}_2^{-1}\left\{\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}}\eta[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2];x,y\right\}=\frac{1}{(xy)^{\frac{1}{2}}}\{H_1[\frac{2}{x}+\frac{2}{y}]+4(\frac{1}{x}+\frac{1}{y})H_2[\frac{2}{x}+\frac{2}{y}]\},$$

where $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Corollary 2. Assume the hypotheses of Theorem 2.1.2(b). Then

$$\mathcal{L}_2^{-1}\left\{\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^5}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}}\zeta[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2];x,y\right\}$$
$$=\frac{1}{(xy)^{\frac{1}{2}}}\{\frac{3}{2}H_1[\frac{2}{x}+\frac{2}{y}]+4(\frac{1}{x}+\frac{1}{y})H_2[\frac{2}{x}+\frac{2}{y}]+8(\frac{1}{x}+\frac{1}{y})^2H_3[\frac{2}{x}+\frac{2}{y}]\},$$

where $\Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

2.1.2.3. Example Based Upon Corollaries 1 and 2

Example 2.2.1. Suppose that, in Corollaries 1 and 2 $f(x)=x^{\frac{\alpha}{4}}\ {}_pF_q\left[\begin{matrix}(a)_p\ ;\\(b)_q\ ;\end{matrix}lx\right]$. Then the results in Example 2.2(ii) read

$$\mathcal{L}_2^{-1}\left\{\frac{1}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\alpha+1}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p,\frac{\alpha+4}{4},\frac{\alpha+6}{4},\frac{\alpha+7}{4},\frac{\alpha+9}{4};\\(b)_q\qquad\qquad\qquad\qquad\qquad\quad ;\end{matrix}\frac{4l}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}\right];x,y\right\}$$

$$=\frac{(xy)^{\frac{\alpha+3}{2}}}{2\pi^{\frac{1}{2}}\Gamma(\frac{\alpha+7}{2})(x+y)^{\frac{\alpha+4}{2}}}\left\{{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{\alpha+2}{2},\frac{\alpha+3}{2};\\(b)_q\qquad\qquad\quad ;\end{matrix}l(\frac{xy}{x+y})^2\right]+8(\alpha+1)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{\alpha+3}{2},\frac{\alpha+4}{2};\\(b)_q\qquad\qquad\quad ;\end{matrix}l(\frac{xy}{x+y})^2\right]\right\},$$

where $p \leq q-3$, $\Re\, \alpha > -1$, $\Re\, [s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0$ if $p \leq q-4$, and $\Re\, [s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}} + 2l\cos \pi r] > 0$ (r = 0, 1) if $p = q-3$.

$$\mathcal{L}_2^{-1}\left\{ \frac{1}{s_1^{\frac{1}{2}} s_2^{\frac{1}{2}} (s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}})^{\alpha+1}}\ {}_{p+4}F_q\left[\begin{matrix}(a)_p, \frac{\alpha+4}{4}, \frac{\alpha+6}{4}, \frac{\alpha+9}{4}, \frac{\alpha+11}{4}; \\ (b)_q \end{matrix} \begin{matrix} \frac{4l}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4} \\ ; \end{matrix}\right]; \bar{x} \right\}$$

$$= \frac{(xy)^{\frac{\alpha+3}{2}}}{2\pi^{\frac{1}{2}}\Gamma(\frac{\alpha+9}{2})(x+y)^{\frac{\alpha+4}{2}}} \left\{ \frac{3}{2}\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{\alpha+2}{2}, \frac{\alpha+3}{2}; \\ (b)_q \quad ; \end{matrix}\ l(\tfrac{xy}{x+y})^2\right] + 4(\alpha+1)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{\alpha+3}{2}, \frac{\alpha+4}{2}; \\ (b)_q \quad ; \end{matrix}\ l(\tfrac{xy}{x+y})^2\right] \right.$$

$$\left. +4(\alpha+1)(2\alpha+3)\ {}_{p+2}F_q\left[\begin{matrix}(a)_p, \frac{\alpha+4}{2}, \frac{\alpha+5}{2}; \\ (b)_q \quad ; \end{matrix}\ l(\tfrac{xy}{x+y})^2\right]\right\},$$ where $p \leq q-3$, $\Re\, \alpha > -1$,

$\Re\, [s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0$ if $p \leq q-4$, and $\Re\, [s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}} + 2l\cos r\pi] > 0$ (r = 0, 1) if $p = q-3$.

Theorem 2.1.3. Assume that $f(x^2), s^{-\frac{1}{2}}\phi(\frac{1}{x})$ and $x^{-\frac{3}{2}}\xi(\frac{1}{x^2})$ are functions of class Ω.

Let **(*i*)** $\mathcal{L}\{f(x); s\} = \phi(s)$,

(*ii*) $\mathcal{L}\{x^{-\frac{1}{2}}\phi(\frac{1}{x}); s\} = \xi(s)$,

(*iii*) $\mathcal{L}\{x^{-\frac{3}{2}}\xi(\frac{1}{x^2}); s\} = \theta(s)$,

(*iv*) $\mathcal{L}\{xf(x^2); s\} = H(s)$,

where $x^{-\frac{1}{2}}\exp(-sx - \frac{u}{x})f(u)$ and $x^{-\frac{1}{2}}\exp(-sx - \frac{2u^{\frac{1}{2}}}{x})f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$.

Then

$$\mathcal{L}_n^{-1}\left\{ \frac{p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]; \bar{x} \right\} = \frac{2}{\pi^{\frac{n-2}{2}} p_n(\overline{x^{\frac{1}{2}}})} H[2p_1(\overline{x^{-1}})]. \tag{1.3.1}$$

It is assumed that the integrals involved exist for n = 2, 3, ..., N.

Proof: First we apply the definition of Laplace transform to (i) and we obtain

$\phi(s) = \int_0^\infty \exp(-st)f(t)dt$, $\Re\, s > c_0$ where c_0 is a constant.

So that

$$\int_0^\infty x^{-\frac{1}{2}} \exp(-sx)\phi(\tfrac{1}{x})dx = \int_0^\infty [\int_0^\infty x^{-\frac{1}{2}} \exp(-sx - \tfrac{u}{x}) f(u)du]dx \tag{1.3.2}$$

The integrand $x^{-\frac{1}{2}} \exp(-sx - \frac{u}{x}) f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, so, by Fubini's Theorem, interchanging the order of the integral on the right side of (1.3.2) is permissible. By using (ii) on the left side and interchanging the order of integration on the right side of (1.3.2), we obtain

$$\xi(s) = \int_0^\infty f(u)[\int_0^\infty x^{-\frac{1}{2}} \exp(-sx - \tfrac{u}{x})dx]du, \text{ where } \Re e\, s > c_1 \text{ and } c_1 \text{ is a constant.} \tag{1.3.3}$$

From the tables by Roberts and Kaufman [87], (1.3.3) reads

$$\xi(s) = (\tfrac{\pi}{s})^{\frac{1}{2}} \int_0^\infty f(u) \exp(-2u^{\frac{1}{2}} s^{\frac{1}{2}})du. \tag{1.3.4}$$

Plugging (1.3.4) in (*iii*), we arrive at

$$\theta(s) = \pi^{\frac{1}{2}} \int_0^\infty [\int_0^\infty x^{-\frac{1}{2}} f(u) \exp(-sx - \tfrac{2u^{\frac{1}{2}}}{x})du]dx, \ \Re e\, s > c_2 \text{ and } c_2 \text{ is a constant.} \tag{1.3.5}$$

By hypothesis, $x^{-\frac{1}{2}} f(u) \exp(-sx - \frac{2u^{\frac{1}{2}}}{x})$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Using Fubini's Theorem, we interchange the order of integration on the right side of (1.3..5). We then use a well known result in Roberts and Kaufman [87] on the resulting integral on the right hand side of this relation to obtain

$$\theta(s) = \pi \int_0^\infty s^{-\frac{1}{2}} f(u) \exp(-2^{\frac{3}{2}} u^{\frac{1}{4}} s^{\frac{1}{2}})du, \ \Re e\, s > c_2. \tag{1.3.6}$$

By substituting $u = v^2$ in (1.3.6), we obtain

$$s^{\frac{1}{2}} \theta(s) = 2\pi \int_0^\infty v f(v^2) \exp(-2^{\frac{3}{2}} v^{\frac{1}{2}} s^{\frac{1}{2}})dv, \ \Re e\, s > c_2. \tag{1.3.7}$$

Next, we replace s with $[p_1(\overline{s^{\frac{1}{2}}})]^2$ and we multiply both sides of (1.3.7) by $p_n(\overline{s^{\frac{1}{2}}})$, we arrive at

$$p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]=2\pi\int_0^\infty vf(v^2)p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}v^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}}))dv. \tag{1.3.8}$$

Using the following known result from Ditkin and Prudnikov [43]:

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}}) \doteqdot (\pi x_i)^{-\frac{1}{2}}\exp(-\tfrac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, \text{n}.$$

Equation (1.3.8) can be rewritten as

$$p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]\overset{n}{=}\frac{2}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_0^\infty vf(v^2)\exp(-2vp_1(\overline{x^{-1}}))dv. \tag{1.3.9}$$

Using (iv) in (1.3.9), we obtain

$$p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2]\overset{n}{=}\frac{2}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}H[2p_1(\overline{x^{-1}})].$$

Therefore,

$$\mathcal{L}_n^{-1}\left\{\frac{p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\theta[(p_1(\overline{s^{\frac{1}{2}}}))^2];\bar{x}\right\}=\frac{2}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}H[2p_1(\overline{x^{-1}})].$$

2.1.3.1. Applications of Theorem 2.1.3

Regarding the application of Theorem 2.1.3, we provide a few examples demonstrating ways of obtaining new N-dimensional transform pairs.

Example 3.1. Suppose that $f(x)=(\pi x)^{-\frac{1}{2}}\cos[\frac{2}{a}x^{\frac{1}{2}}]$. Then

$$\phi(s)=s^{-\frac{1}{2}}\exp(-\tfrac{1}{a^2 s}),\ \Re e\, s>0.$$

$$\xi(s)=\frac{1}{s+\frac{1}{a^2}},\ \Re e\, s>-\frac{1}{a^2}.$$

$$\theta(s)=\frac{a^2p^{\frac{1}{2}}}{2}s^{\frac{1}{2}}S_{-1,\frac{1}{2}}(as),\ \Re e\, a>0,\ \Re e\, s>0.$$

$$H(s)=\frac{\pi s}{s^2+(\frac{2}{a})^2},\ \Re e\, s>2\left|\operatorname{Im}\tfrac{1}{a}\right|.$$

Using (1.3.1), we arrive at

$$\mathcal{L}_n^{-1}\left\{\frac{p_1^{\,2}(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}S_{-1,\frac{1}{2}}\left[a(p_1(\overline{s^{\frac{1}{2}}}))^2\right];\bar{x}\right\}=\frac{2p_1(\overline{x^{-1}})}{\pi^{\frac{n-3}{2}}p_n(\overline{x^{\frac{1}{2}}})[a^2p_1^{\,2}(\overline{x^{-1}})+1]},$$

where $\Re e\, a>0$, $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})]>0$, $n=2,3,\cdots N$. (1.1)

Example 3.2. Let $f(x)=x^{-\alpha}G_{h,k}^{m,n}\left(x\middle|{a_1,\ldots,a_h \atop b_1,\ldots,b_k}\right)$, where $h+k<2(m+n)$, $\Re e\,\alpha>\Re e\, b_j+1,\ j=1,\ldots,m$. (The same formula is valid if $h<k$ (or $h=k$ and $\Re e\, s>1$) and $\Re e\,\alpha<\Re e\, b_j+1,\ j=1,2,\ldots,m$.)

Then

$$\phi(s)=s^{\alpha-1}G_{h+1,k}^{m,n+1}\left(\tfrac{1}{s}\middle|{\alpha,a_1,\ldots,a_h \atop b_1,\ldots,b_k}\right),\ |\arg s|<(m+n-\tfrac{1}{2}h-\tfrac{1}{2}k)\pi.$$

$$\xi(s)=s^{\alpha-\frac{3}{2}}G_{h+2,k}^{m,n+2}\left(\tfrac{1}{s}\middle|{\alpha-\frac{1}{2},\alpha,a_1,\ldots,a_h \atop b_1,\ldots,b_k}\right),\ |\arg s|<(m+n-\tfrac{1}{2}h-\tfrac{1}{2}k+\tfrac{1}{2})\pi.$$

$$\theta(s)=\pi^{-\frac{1}{2}}2^{-\alpha}s^{\alpha-1}G_{h+4,k}^{m,n+4}\left(\tfrac{4}{s^2}\middle|{\alpha-\frac{1}{2},\alpha,\frac{\alpha}{2},\frac{\alpha+1}{2},a_1,\ldots,a_h \atop b_1,\ldots,b_k}\right),$$

where $\Re e\,(1-\alpha)+2\min\limits_j \Re e\, b_j>0$ $(j=1,2,\ldots,k)$, $\Re e\, s>0$.

$$H(s)=\pi^{-\frac{1}{2}}2^{(1-2\alpha)}s^{(2\alpha-2)}G_{h+2,k}^{m,n+2}\left(\tfrac{4}{s^2}\middle|{\frac{2\alpha-1}{2},\alpha,a_1,\ldots,a_h \atop b_1,\ldots,b_k}\right),\text{where}$$

$\Re e\,(2-2\alpha)+2\min\limits_j \Re e\, b_j>0$ $(j=1,2,\ldots,k)$, $\Re e\, s>0$.

Now we apply Theorem 2.1.3, to obtain

$$\mathcal{L}_n^{-1}\left\{\frac{[p_1(\overline{s}^{\frac{1}{2}})]^{2\alpha-1}}{p_n(\overline{s}^{\frac{1}{2}})} G_{h+4,k}^{m,n+4}\left(\frac{4}{p_1^4(\overline{s}^{\frac{1}{2}})}\middle|\begin{matrix}\alpha-\frac{1}{2},\alpha,\frac{\alpha}{2},\frac{\alpha+1}{2},a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right);\overline{x}\right\}$$

$$=\frac{2^{\alpha}}{\pi^{\frac{n-2}{2}}\,p_n(\overline{x}^{\frac{1}{2}})} G_{h+2,k}^{m,n+2}\left(\frac{1}{p_1^2(\overline{x}^{-1})}\middle|\begin{matrix}\alpha-\frac{1}{2},\alpha,a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right), \tag{2.1}$$

where $\Re e\ \alpha < 1+2\min_j \Re e\ b_j\ (j=1,2,\dots,k)$, $\Re e\ [p_1(\overline{s}^{\frac{1}{2}})] > 0$.

Corollary 1. Let n = 2 and assume the hypotheses of Theorem 2.1.3. Then

$$\mathcal{L}_2^{-1}\left\{\frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}}\,\theta[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2];x,y\right\} = \frac{2}{(xy)^{\frac{1}{2}}}H[\tfrac{2}{x}+\tfrac{2}{y}],\quad \Re e\ (s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})>0.$$

2.1.3.2. Examples Based Upon Corollary 1

Example 3.1'

Assume that $f(x)=(\pi x)^{-\frac{1}{2}}\cos(2x^{\frac{1}{2}})$. Then from Corollary 1 and Example 3.1, we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}{(s_1 s_2)^{\frac{1}{2}}}\,\underset{-1,\frac{1}{2}}{S}[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2];x,y\right\} = \frac{2\pi^{\frac{1}{2}}(xy)^{\frac{1}{2}}(x+y)}{(x+y)^2+x^2y^2}, \tag{1.1'}$$

where $\Re e\ (s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})>0$.

Example 3.2'

In Example 3.2 the Inverse Laplace transform (2.1) in two dimensions reads

$$\mathcal{L}_2^{-1}\left\{\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\alpha-1}}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}}G_{h+4,k}^{m,n+4}\left(\frac{4}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^4}\Big|\begin{matrix}\alpha-\frac{1}{2},\alpha,\frac{\alpha}{2},\frac{\alpha+1}{2},a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right);x,y\right\}$$

$$=\frac{2^{\alpha}}{(xy)^{\frac{1}{2}}}G_{h+2,k}^{m,n+2}\left(\left(\frac{xy}{x+y}\right)^2\Big|\begin{matrix}\alpha-\frac{1}{2},\alpha,a_1,\dots,a_h\\ b_1,\dots,b_k\end{matrix}\right), \tag{2.1'}$$

where $\Re e\ \alpha<1+2\min_j \Re e\ b_j\ (j=1,2,\dots,k)$, $\Re e\ (s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})>0$.

Remark 2.1.2.1: The formulas (1.1')and (2.1') both are new formulas for calculating the Inverse Laplace transformations in two dimensions.

2.2. The Original of Functions with the Argument $[p_1(s^{\overline{\frac{1}{2}}})]^{-1}$

Our considerations in this section will center on the inverse Laplace transformations of functions of the form $\frac{\gamma[(p_1(s^{\overline{\frac{1}{2}}}))^{-1}]}{[p_1(s^{\overline{\frac{1}{2}}})]^{\frac{3}{2}}}$ and $\frac{\wedge[(p_1(s^{\overline{\frac{1}{2}}}))^{-1}]}{[p_1(s^{\overline{\frac{1}{2}}})]^{\upsilon+1}}$.

The results obtained in Theorems 2.2.1 and 2.2.2, are easily adaptable for obtaining two dimensional inverse Laplace transform pairs. We have, therefore, given several new inverse Laplace transformations.

Theorem 2.2.1. Suppose that f is a function of class Ω.

If

(i) $\mathcal{L}\ \{f(x);s\}=\phi(s)$,

(ii) $-\frac{d}{ds}\{s^{-\frac{1}{2}}\phi(\frac{1}{s})\}=\gamma(s)$,

(iii) $\mathcal{L}\ \{x^{\frac{n-1}{2}}f(2x^{\frac{1}{2}});s\}=G(s)$,

(iv) $\mathcal{L}\ \{x^{\frac{n+1}{2}}f(2x^{\frac{1}{2}});s\}=H(s)$,

and suppose that $\frac{d}{ds}\{s^{-\frac{1}{2}}\phi(\frac{1}{2})\}$ exists for $\Re e\,(s) > c_o$, where c_0 is a constant.

Then

$$\mathcal{L}_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}}\gamma[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}];\bar{x}\right\} = \frac{1}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{3}{2}}})}\{(n+\tfrac{1}{2})G[p_1(\overline{x^{-1}})] - 2p_1(\overline{x^{-1}})H[p_1(\overline{x^{-1}})]\}, \qquad (1.1)$$

where n = 2, 3, ..., N. It is assumed that the integrals involved on the left exist. The conditions for two dimensions are given in Ditkin and Prudnikov [43; p. 32]. In general, for n variables, we have the same result under analogous assumptions.

Proof: First we apply the definition of Laplace transform to *(i)*, to obtain

$$\phi(s) = \int_o^\infty \exp(-st)f(t)dt,\ \Re e\, s > c_o, \qquad (1.2)$$

From (1.2) and *(ii)*, we obtain

$$s^{\frac{3}{2}}\gamma(s) = \frac{1}{2}\phi(\frac{1}{s}) + \frac{1}{s}\frac{d}{d(\frac{1}{s})}\phi(\frac{1}{s}) = \int_o^\infty (\frac{1}{2} - \frac{u}{s})\exp(-\frac{u}{s})f(u)du. \qquad (1.3)$$

Replacing s with $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and multiplying both sides of (1.3) by $p_n(\bar{s})$, we obtain

$$p_n(\bar{s})\left[p_1(\overline{s^{\frac{1}{2}}})\right]^{-\frac{3}{2}}\gamma\left[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}\right] = \int_0^\infty \left[\frac{1}{2}p_n(\bar{s}) - up_1(\overline{s^{\frac{1}{2}}})p_n(\bar{s})\right]$$
$$\cdot \exp(-up_1(\overline{s^{\frac{1}{2}}}))f(u)du. \qquad (1.4)$$

Next we use the following two well-known operational results in Ditkin and Prudnikov [43]

$$s_i\exp(-as_i^{\frac{1}{2}}) \doteqdot \frac{a}{2\pi^{\frac{1}{2}}}x_i^{-\frac{3}{2}}\exp\left(-\frac{a^2}{4x_i}\right),\ \text{for } i = 1,2,\ldots,n,$$

$$s_i^{\frac{3}{2}} \exp(-as_i^{\frac{1}{2}}) \doteqdot \frac{a^2-2x_i}{4\pi^{\frac{1}{2}}} x_i^{-\frac{5}{2}} \exp\left(-\frac{a^2}{4x_i}\right), \text{ for } i=1,2,\ldots,n,$$

to obtain

$$p_n(\overline{s}) \left[p_1(\overline{s^{\frac{1}{2}}})\right]^{-\frac{3}{2}} \gamma\left[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}\right] \underset{\overline{n}}{\overset{n}{=}} \frac{p_n(\overline{x^{-\frac{3}{2}}})}{2^{n+1}\pi^{\frac{n}{2}}} \tag{1.5}$$

$$\left\{(1+2n)\int_o^\infty u^n \exp\left(-\frac{u^2}{4} p_1(\overline{x^{-1}})\right) f(u)du - p_1(\overline{x^{-1}})\int_o^\infty u^{n+2} \exp\left(-\frac{u^2}{4} p_1(\overline{x^{-1}})\right) f(u)du\right\}.$$

Substituting $v=\dfrac{u^2}{4}$ and using (*iii*) and (*iv*) into (1.5), we arrive at

$$p_n(\overline{s})\left[p_1(\overline{s^{\frac{1}{2}}})\right]^{-\frac{3}{2}} \gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right] \underset{\overline{n}}{\overset{n}{=}} \frac{p_n(\overline{x^{-\frac{3}{2}}})}{\pi^{\frac{n}{2}}}$$

$$\cdot\left\{\left(n+\frac{1}{2}\right)G\left[p_1(\overline{x^{-1}})\right] - 2p_1(\overline{x^{-1}})H\left[p_1(\overline{x^{-1}})\right]\right\}.$$

Hence

$$L_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}} \gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]; \overline{x}\right\}$$

$$= \frac{1}{\pi^{\frac{n}{2}} p_n(\overline{x^{\frac{3}{2}}})}\left\{\left(n+\frac{1}{2}\right)G\left[p_1(\overline{x^{-1}})\right] - 2p_1(\overline{x^{-1}})H\left[p_1(\overline{x^{-1}})\right]\right\},$$

where n = 2, 3, ..., N.

2.2.1. Applications of Theorem 2.2.1

We now present some direct applications of the Theorem 2.2.1. Initially we consider applications on some functions of n variables and obtain their inverse Laplace transforms in n variables. As these functions and their transform functions are usually very complex in nature, we will also consider functions having only two variables and their corresponding transforms.

Example 2.2.1. The functions used for $f(x)$ in this example are $x^{\alpha}\exp(-ax)$, $\Re e\ \alpha>-1$, $\ln x$, $H_{2n}(x)$ and $x^{\alpha}{}_pF_q\left[\begin{matrix}(a)_p;\\(b)_q;\end{matrix}(kx)^2\right]$. Then using Theorem 2.2.1, the inverse Laplace transformations obtained are given in N-dimensions in the respective order.

$$L_n^{-1}\left\{\left(p_1(\overline{s^{\frac12}})\right)^{-\alpha-1}\left[1+a\left(p_1(\overline{s^{\frac12}})\right)^{-1}\right]^{-\alpha-2}\left[\frac12 a\left(p_1(\overline{s^{\frac12}})\right)^{-1}-\alpha-\frac12\right];\bar{x}\right\}$$

$$=\frac{2^{\frac{\alpha-n+1}{2}}\Gamma(\alpha+n+1)p_n(\overline{x^{-\frac32}})}{\pi^{\frac n2}\Gamma(\alpha+1)\left[p_1(\overline{x^{-1}})\right]^{\frac{\alpha+n+1}{2}}}\exp\left(\frac{a^2}{2p_1(\overline{x^{-1}})}\right)$$

$$\cdot\left\{\left(n+\frac12\right)D_{-\alpha-n-1}\left[\left(\frac{2a^2}{p_1(\overline{x^{-1}})}\right)^{\frac12}\right]-\frac{(\alpha+n+1)(\alpha+n+2)}{2}D_{-\alpha-n-3}\left[\left(\frac{2a^2}{p_1(\overline{x^{-1}})}\right)^{\frac12}\right]\right\},\qquad(1.1)$$

where $\Re e\ \alpha>-n-1$, $\Re e\ a>0$, $\Re e\left[p_1(\overline{s^{\frac12}})\right]>-\Re e\ a$, n = 2, 3, ..., N.

$$L_n^{-1}\left\{\frac{1}{p_1(\overline{s^{\frac12}})}\left[\ln\gamma p_1(\overline{s^{\frac12}})-2\right];\bar{x}\right\}$$

$$=\frac{2\Gamma(\frac{n+1}{2})}{\pi^{\frac n2}p_n(\overline{x^{\frac32}})\left[p_1(\overline{x^{-1}})\right]^{\frac{n+1}{2}}}\left\{\ln[p_1(\overline{x^{-1}})]-\tfrac12\psi(\tfrac{n+1}{2})-\tfrac{2}{n+1}-\ln 2\right\},\qquad(1.2)$$

where $\Re e\left[p_1(\overline{s^{\frac12}})\right]>0$, n = 2, 3, ..., N.

$$L_n^{-1}\left\{\frac{1}{p_1(\overline{s^{\frac12}})}\left\{{}_2F_0\left[-n,1;\frac{2}{[p_1(\overline{s^{\frac12}})]^2}\right]-\frac{8n}{[p_1(\overline{s^{\frac12}})]^2}{}_2F_0\left[-n+1;2;\frac{2}{[p_1(\overline{s^{\frac12}})]^2}\right]\right\},\bar{x}\right\}$$

$$=\frac{2^{n+2}\Gamma(\frac{n+1}{2})}{\pi^{\frac n2}p_n(\overline{x^{\frac32}})[p_1(\overline{x^{-1}})]^{\frac{n+1}{2}}}\left\{(n+1){}_2F_1\left[\begin{matrix}-n,\frac{n+3}{2};\\ \frac12\end{matrix};\frac{4}{p_1(\overline{x^{-1}})}\right]-(n+\tfrac12)\,{}_2F_1\left[\begin{matrix}-n,\frac{n+1}{2};\\ \frac12\end{matrix};\frac{4}{p_1(\overline{x^{-1}})}\right]\right\},\qquad(1.3)$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 4$, $n = 2, 3, \ldots, N$.

$$\mathcal{L}_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^{\alpha+1}}\left\{\left(\alpha-\frac{1}{2}\right)\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,\dfrac{\alpha+1}{2},\dfrac{\alpha+2}{2};\dfrac{4k^2}{}\\(b)_q\qquad\qquad ;[p_1(\overline{s^{\frac{1}{2}}})]^2\end{matrix}\right]\right.\right.$$
$$\left.\left.+\frac{2k^2(\alpha+1)(\alpha+2)\prod_{m=1}^{p}a_m}{\prod_{j=1}^{q}b_j}\cdot\frac{1}{p_1(\overline{s^{\frac{1}{2}}})}\,{}_{p+2}F_q\left[\begin{matrix}(a+1)_p,\dfrac{\alpha+3}{2},\dfrac{\alpha+4}{2};\dfrac{4k^2}{}\\(b+1)_q\qquad\qquad ;[p_1(\overline{s^{\frac{1}{2}}})]^2\end{matrix}\right]\right\};\overline{x}\right\}$$
$$=\frac{2^{\alpha}\Gamma(\frac{n+\alpha+1}{2})p_n(\overline{x^{-\frac{3}{2}}})}{\pi^{\frac{n}{2}}\Gamma(\alpha+1)[p_1(\overline{x^{-1}})]^{\frac{n+\alpha+1}{2}}}$$
$$\left\{(n+\alpha+1)\,{}_{p+1}F_q\left[\begin{matrix}(a)_p,\dfrac{n+\alpha+1}{2};\dfrac{4k^2}{}\\(b)_q\qquad ;p_1(\overline{x^{-1}})\end{matrix}\right]-\left(n+\frac{1}{2}\right)\,{}_{p+1}F_q\left[\begin{matrix}(a)_p,\dfrac{n+\alpha+3}{2};\dfrac{4k^2}{}\\(b)_q\qquad ;p_1(\overline{x^{-1}})\end{matrix}\right]\right\},$$

where $p \le q-1$, $\Re e\,\alpha > -1$, $n = 2, 3, \ldots, N$,
$\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$ if $p \le q-2$, $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})+2k\cos pr] > 0$ $(r=0,1)$, if $p=q-1$
$\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > \Re e\,(4k^2)$ if $p=q$. (1.4)

Example 2.2.1'. We consider the case n = 2 in this example.
Substituting $n = 2$, $\alpha = -\frac{1}{2}$ and $a = 1$ in (1.1), we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{1}{1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\frac{3}{2}}};x,y\right\}$$
$$=\frac{5}{2^{\frac{11}{4}}(xy)^{\frac{1}{4}}(x+y)^{\frac{5}{4}}}\exp\left(\frac{xy}{2(x+y)}\right)\left\{D_{-\frac{5}{2}}\left[\left(\frac{2xy}{x+y}\right)^{\frac{1}{2}}\right]-\frac{7}{8}D_{-\frac{9}{2}}\left[\left(\frac{2xy}{x+y}\right)^{\frac{1}{2}}\right]\right\},$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > -1$ (1.1')

Equation (1.2) in two dimensions reduces to the following result.

$$\mathcal{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}\ln\gamma(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}});x,y\right\}$$

$$= \frac{1}{\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\left[\frac{\gamma-1}{2}+\ln\frac{x+y}{xy}\right], \text{ where } \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \qquad (1.2')$$

If we take $n = 2$, then (1.3) leads to the following result.

$$\mathfrak{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}\left\{{}_0F_2\left[-2,1;\frac{2}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right]\frac{16}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}{}_0F_2\left[-1,2;\frac{2}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right]\right\};x,y\right\}$$

$$= \frac{4}{\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\left\{6\;{}_2F_1\left[\begin{matrix}-2,\frac{5}{2}; & \frac{4xy}{}\\ \frac{1}{2} & ;x+y\end{matrix}\right]-5\;{}_2F_1\left[\begin{matrix}-2,\frac{3}{2}; & \frac{4xy}{}\\ \frac{1}{2} & ;x+y\end{matrix}\right]\right\}, \text{ where } \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>4. \qquad (1.3')$$

Taking n = 2, $\alpha = \frac{1}{2}$ and $k = \frac{1}{2}$ in (1.4), we arrive at

$$\mathfrak{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\frac{3}{2}}}\left\{{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{3}{4},\frac{5}{4} & ;\frac{1}{}\\ (b)_q & ;(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\end{matrix}\right]\right.\right.$$

$$\left.\left.+\frac{15\prod_{m=1}^{p}a_m}{8\prod_{j=1}^{q}b_j}\cdot\frac{1}{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a+1)_p,\frac{7}{4},\frac{9}{4} & ;\frac{1}{}\\ (b+1)_q & ;(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\end{matrix}\right]\right\};\bar{x}\right\}$$

$$= \frac{3(xy)^{\frac{1}{4}}}{4\pi^{\frac{1}{2}}\Gamma(\frac{1}{4})(x+y)^{\frac{7}{4}}}\left\{7\;{}_{p+1}F_q\left[\begin{matrix}(a)_p,\frac{7}{4}; & \frac{xy}{}\\ (b)_q & ;x+y\end{matrix}\right]-5\;{}_{p+1}F_q\left[\begin{matrix}(a)_p,\frac{11}{4}; & \frac{xy}{}\\ (b)_q & ;x+y\end{matrix}\right]\right\},$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$ if $p \le q-2$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+\cos\pi r]>0$ $(r = 0, 1)$ if $p = q-1$, $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1$ if $p = q$. (1.4')

Remark 2.2.1: All results which we have derived in Example 2.2.1', are new formulas for calculating two dimensional inverse Laplace transforms of corresponding functions.

The following Theorem is a generalization of the Theorem 2.2.1.

Theorem 2.2.2. Suppose that f is function of class Ω.

If

(i) $\mathcal{L}\left\{f(x);s\right\} = \phi(s)$,

(ii) $-\dfrac{d}{ds}\left\{s^{-\nu}\phi(\frac{1}{s})\right\} = \Lambda(s)$,

(iii) $\mathcal{L}\left\{x^{\frac{n-1}{2}}f(2x^{\frac{1}{2}});s\right\} = G(s)$,

(iv) $\mathcal{L}\left\{x^{\frac{n+1}{2}}f(2x^{\frac{1}{2}});s\right\} = H(s)$,

and assume that $\frac{d}{ds}\{s^{-\nu}\phi(\frac{1}{s})\}$ exists for $\Re e\,(s) > c_o$, where c_o is a constant.

Then

$$\mathcal{L}_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\Lambda[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}];\bar{x}\right\}$$

$$= \frac{1}{\pi^{\frac{n}{2}}p_n(x^{\frac{3}{2}})}\left\{(\nu+n)G[p_1(\overline{x^{-1}})] - 2p_1(\overline{x^{-1}})H[p_1(\overline{x^{-1}})]\right\},$$

provided the integrals involved on the left side of (2.1) exist for $n = 2, 3, \ldots, N$. The proof of this Theorem is similar to that of Theorem 2.2.1, so that, we prove Theorem 2.2.2 in brief.

Proof: Using (*i*) and (*ii*), we arrive at

$$s^{\nu+1}\Lambda(s) = \int_0^\infty [\nu - \tfrac{u}{s}]\exp\left(-\tfrac{u}{s}\right)f(u)du \tag{2.2}$$

Replace s with $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and multiply both sides of (2.2) by $p_n(\bar{s})$, then using the two known operational results from Ditkin and Prudnikov [43] which we have used in the proof of Theorem 2.2.1, and then making a change of variable $\upsilon = \frac{u^2}{4}$, we obtain

$$p_n(\bar{s})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\Lambda[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}] \stackrel{n}{\underset{n}{=}} \frac{p_n(\overline{x^{-\frac{3}{2}}})}{\pi^{\frac{n}{2}}}\{(\tau+n)$$

$$\cdot\int_0^\infty \upsilon^{\frac{n-1}{2}}\exp(-\upsilon p_1(\overline{x^{-1}}))f(2\upsilon^{\frac{1}{2}})d\upsilon - 2p_1(\overline{x^{-1}})\int_0^\infty \upsilon^{\frac{n+1}{2}}\exp(-\upsilon p_1(\overline{x^{-1}}))f(2\upsilon^{\frac{1}{2}})d\upsilon\}. \tag{2.3}$$

Hence by using (*iii*) and (*iv*), we arrive at

$$\mathfrak{L}_n^{-1}\left\{\frac{1}{[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\Lambda\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right];\bar{x}\right\}=\frac{1}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{3}{2}}})}\left\{(\nu+n)G\left[p_1(\overline{x^{-1}})\right]-2p_1(\overline{x^{-1}})H\left[p_1(\overline{x^{-1}})\right]\right\}.$$

2.2.2. Applications of Theorem 2.2.2

We can now conveniently apply Theorem 2.2.2 to derive the N-dimensional inverse Laplace transforms for some functions with n-variables. These results are discussed as follows:

Example 2.2.2. Consider $f(x) = J_0(ax)$, then

$$\phi(s)=\frac{1}{(s^2+a^2)^{\frac{1}{2}}},\ \Re e\, s>|\mathrm{Im}\,a|.$$

$$\Lambda(s)=\frac{(\nu-1)+\nu a^2 s^2}{s^{\nu}(1+a^2 s^2)^{\frac{3}{2}}},\ \Re e\, s>|\mathrm{Im}\,a|.$$

$$G(s)=\frac{\Gamma(\frac{n+1}{2})}{s^{\frac{n+1}{2}}}\ {}_1F_1\left[\frac{n+1}{2};1;-\frac{a^2}{s}\right],\ \Re e\, s>0.$$

$$H(s)=\frac{\Gamma(\frac{n+3}{2})}{s^{\frac{n+3}{2}}}\ {}_1F_1\left[\frac{n+3}{2};1;-\frac{a^2}{s}\right],\ \Re e\, s>0.$$

Plugging in (2.1), we obtain

$$\mathfrak{L}_n^{-1}\left\{\frac{\nu\left[(p_1(\overline{s^{\frac{1}{2}}}))^2+a^2\right]-\left[p_1(\overline{s^{\frac{1}{2}}})\right]^2}{\left[(p_1(\overline{s^{\frac{1}{2}}}))^2+a^2\right]^{\frac{3}{2}}};\bar{x}\right\}$$

$$= \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n}{2}}} \cdot \frac{1}{p_n(\overline{x^{\frac{3}{2}}})[p_1(\overline{x^{-1}})]^{\frac{n+1}{2}}}$$

$$\left\{(\nu+n)\,{}_1F_1\left[\frac{n+1}{2};1;-\frac{a^2}{p_1(\overline{x^{-1}})}\right]-(n+1)\,{}_1F_1\left[\frac{n+3}{2};1;-\frac{a^2}{p_1(\overline{x^{-1}})}\right]\right\},$$

where n = 2, 3, 4, ..., N, $\Re e\left[p_1(\overline{s^{\frac{1}{2}}})\right]>|\mathrm{Im}a|$. (2.1.1)

Example 2.2.3. Let $f(x) = xJ_1(ax)$, then

$$\phi(s)=\frac{a}{(s^2+a^2)^{\frac{3}{2}}}, \Re e\, s>|\mathrm{Im}a|.$$

$$\Lambda(s)=\frac{as^{-\nu+2}\left[(\nu-3)+\nu a^2s^2\right]}{(1+a^2s^2)^{\frac{5}{2}}},\ \Re e\, s>|\mathrm{Im}\, a|.$$

$$G(s)=\frac{2a\Gamma(\frac{n+3}{2})}{s^{\frac{n+3}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{n+3}{2} & ; -\frac{a^2}{s}\\ 2 & ;\end{matrix}\right].$$

$$H(s)=\frac{2a\Gamma(\frac{n+5}{2})}{s^{\frac{n+5}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{n+5}{2} & ; -\frac{a^2}{s}\\ 2 & ;\end{matrix}\right],\ \Re e\, s>0.$$

Hence from (2.1), we arrive at

$$\mathcal{L}_n^{-1}\left\{\frac{(\nu-3)p_1^{\,2}(\overline{s^{\frac{1}{2}}})+\nu a^2}{\left[p_1^{\,2}(\overline{s^{\frac{1}{2}}})+a^2\right]^{\frac{5}{2}}};x\right\}$$

$$=\frac{2\Gamma(\frac{n+3}{2})}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{3}{2}}})[p_1(\overline{x^{-1}})]^{\frac{n+3}{2}}}\left\{(\nu+n)\,{}_1F_1\left[\begin{matrix}\frac{n+3}{2} & ; -\frac{a^2}{p_1(\overline{x^{-1}})}\\ 2 & ;\end{matrix}\right]-\frac{n+3}{2}\,{}_1F_1\left[\begin{matrix}\frac{n+5}{2} & ; -\frac{a^2}{p_1(\overline{x^{-1}})}\\ 2 & ;\end{matrix}\right]\right\},$$

where $\Re e\left[p_1(\overline{s^{\frac{1}{2}}})\right]>|\mathrm{Im}a|$. (2.1.2)

Example 2.2.4. Assume that $f(x) = {}_0F_2\left[\begin{matrix} & ; \\ \frac{1}{2},1 & ;\end{matrix}(\frac{x}{2})^2\right]$. Then

$$\phi(s)=\frac{1}{s}\exp(s^{-2}), \Re e\, s>0.$$

$$\Lambda(s)=[(\nu-1)s^{-\nu}-2s^{-\nu+2}]\exp(s^2),\ \Re e\, s>0.$$

$$G(s)=\frac{\Gamma(\frac{n+1}{2})}{s^{\frac{n+1}{2}}}\ {}_1F_2\left[\begin{matrix}\frac{n+1}{2} & ; \\ \frac{1}{2},1 & ;\end{matrix}\frac{1}{s}\right],\ \Re e\, s>0.$$

$$H(s)=\frac{\Gamma(\frac{n+3}{2})}{s^{\frac{n+3}{2}}}\ {}_1F_2\left[\begin{matrix}\frac{n+3}{2} & ; \\ \frac{1}{2},1 & ;\end{matrix}\frac{1}{s}\right],\ \Re e\, s>0.$$

Using (2.1), we obtain

$$\mathcal{L}_n^{-1}\left\{\frac{(\nu-1)(p_1{}^2(\overline{s^{\frac{1}{2}}})-2}{p_1{}^3(\overline{s^{\frac{1}{2}}})}\exp[\frac{1}{p_1{}^2(\overline{s^{\frac{1}{2}}})}];\bar{x}\right\}$$

$$=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{3}{2}}})[p_1(\overline{x^{-1}})]^{\frac{n+1}{2}}}\left\{(\nu+n)\ {}_1F_2\left[\begin{matrix}\frac{n+1}{2}; \\ \frac{1}{2},1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]-(n+1)\ {}_1F_2\left[\begin{matrix}\frac{n+3}{2}; \\ \frac{1}{2},1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]\right\},\text{ where}$$

$$\Re e\left[p_1(\overline{s^{\frac{1}{2}}})\right]>0. \tag{2.1.3}$$

From Theorem 2.2.2 we easily obtain the following formula for two dimensions.

Corollary 1. If $n = 2$, then formula (2.1) in Theorem 2.2.2 reduces to the following result.

$$\mathcal{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\frac{3}{2}}}\Lambda[\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}];x,y\right\}$$

$$=\frac{1}{\pi(xy)^{\frac{3}{2}}}\{(\nu+2)G[\frac{1}{x}+\frac{1}{y}]-2(\frac{1}{x}+\frac{1}{y})H[\frac{1}{x}+\frac{1}{y}]\},\ \Re e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0.$$

2.2.3 Examples Based Upon Corollary 1

We provide several examples concerning Corollary 1 and using examples given in (2.2.2). We will use these new results in Chapter 4 to solve the corresponding partial differential equations.

Example 2.2.2'.

(*i*) If we choose a = ν = 1 in Example 2.2.2 and then using Corollary 1, we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{1}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+1]^{\frac{3}{2}}};x,y\right\}$$
$$=\frac{3}{2\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\left\{{}_1F_1\left[\begin{matrix}\frac{3}{2};\\1;\end{matrix}-\frac{xy}{x+y}\right]-{}_1F_1\left[\begin{matrix}\frac{5}{2};\\1;\end{matrix}-\frac{xy}{x+y}\right]\right\}.$$

Because

$${}_1F_1\left[\begin{matrix}\frac{3}{2};\\1;\end{matrix}-\frac{xy}{x+y}\right]-{}_1F_1\left[\begin{matrix}\frac{5}{2};\\1;\end{matrix}-\frac{xy}{x+y}\right]=\frac{xy}{x+y}\,{}_1F_1\left[\begin{matrix}\frac{5}{2};\\2;\end{matrix}-\frac{xy}{x+y}\right]$$

the right side simplifies as

$$\mathcal{L}_2^{-1}\left\{\frac{1}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+1]^{\frac{3}{2}}};x,y\right\}=\frac{3xy}{2\pi^{\frac{1}{2}}(x+y)^{\frac{5}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{5}{2};\\2;\end{matrix}-\frac{xy}{x+y}\right],\ \Re e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0. \qquad (2.2.\text{i})$$

Similarly in three dimensions, we have the following new result.

$$\mathcal{L}_3^{-1}\left\{\frac{1}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+s_3^{\frac{1}{2}})^2+1]^{\frac{3}{2}}};x,y,z\right\}=\frac{4}{\pi^{\frac{3}{2}}}\cdot\frac{(xyz)^{\frac{3}{2}}}{(yz+zx+xy)^3}\,{}_1F_1\left[\begin{matrix}3;\\1;\end{matrix}-\frac{xyz}{yz+zx+xy}\right],$$

where $\Re e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+s_3^{\frac{1}{2}}\right]>0.$

Remark 2.2.2: Using (2.2.i) and the following operational relation

$$\mathcal{L}_2^{-1}\left\{\frac{1}{s_1 s_2}f(s_1,s_2);x,y\right\}=\int_0^x\int_0^y F(\xi,\eta)\,d\xi\,d\eta,$$

where

$$f(s_1,s_2) = \mathfrak{L}_2\ \{F(x,y), s_1, s_2\}.$$

We derive the following well known formula

$$\mathfrak{L}_2^{-1}\left\{\frac{1}{s_1 s_2[(s_1^{\frac{1}{2}} s_2^{\frac{1}{2}})^2+1]^{\frac{3}{2}}};x,y\right\} = \frac{2}{\pi}\frac{(xy)^{\frac{1}{2}}}{x+y}\exp\left(\frac{xy}{x+y}\right),$$

where $\Re e\,[s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0.$ (2.2.i')

(ii) If we choose a =1 and ν = 2 in Example 2.2.2 and making use of (2.2.i), we arrive at

$$\mathfrak{L}_2^{-1}\left\{\frac{1}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+1]^{\frac{1}{2}}};x,y\right\} = \frac{1}{2\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\ {}_1F_1\left[\begin{matrix}\frac{3}{2};\\ 1;\end{matrix} -\frac{xy}{x+y}\right],$$ where $\Re e\,[s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0.$ (2.2.ii)

(iii) Taking ν = 0 and a = 1 in the same example, yields

$$\mathfrak{L}_2^{-1}\left\{\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+1]^{\frac{3}{2}}};x,y\right\} = \frac{1}{2\pi^{\frac{1}{2}}(x+y)^{\frac{5}{2}}}\left\{(x+y)\ {}_1F_1\left[\begin{matrix}\frac{5}{2};\\ 1;\end{matrix} -\frac{xy}{x+y}\right] - 2xy\ {}_1F_1\left[\begin{matrix}\frac{5}{2};\\ 1;\end{matrix} -\frac{xy}{x+y}\right]\right\},$$

where $\Re e\,[s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0.$ (2.2.iii)

Example 2.2.3'.

(i) We choose ν = 3 in Corollary 1 and a = 1. Then we use the result in Example 2.2.3, to obtain the following new result

$$\mathfrak{L}_2^{-1}\left\{\frac{1}{[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+1]^{\frac{5}{2}}};x,y\right\} = \frac{5xy}{4\pi^{\frac{1}{2}}(x+y)^{\frac{7}{2}}}\left\{xy\ {}_1F_1\left[\begin{matrix}\frac{7}{2};\\ 3;\end{matrix} -\frac{xy}{x+y}\right] + (x+y)\ {}_1F_1\left[\begin{matrix}\frac{5}{2};\\ 2;\end{matrix} -\frac{xy}{x+y}\right]\right\},$$

where $\Re e\,[s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0.$ (2.3.i) *(ii)* Similarly, with $\nu = \dfrac{1}{2}$ and a = 1, we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{1-5\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}{\left[\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2+1\right]^{\frac{5}{2}}};x,y\right\}=\frac{15(xy)^2}{2\pi^{\frac{1}{2}}(x+y)^{\frac{7}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{7}{2}; & -\frac{xy}{x+y}\\ 3; & \end{matrix}\right],\ \Re e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0. \quad (2.3.\text{ii})$$

(iii) From *(i)* and *(ii)* we derive

$$\mathcal{L}_2^{-1}\left\{\frac{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}{\left[\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2+1\right]^{\frac{5}{2}}};x,y\right\}=\frac{xy}{4\pi^{\frac{1}{2}}(x+y)^{\frac{7}{2}}}\left\{(x+y)\,{}_1F_1\left[\begin{matrix}\frac{5}{2} & ; & -\frac{xy}{x+y}\\ 2 & ; & \end{matrix}\right]-5xy\,{}_1F_1\left[\begin{matrix}\frac{7}{2} & ; & -\frac{xy}{x+y}\\ 3 & ; & \end{matrix}\right]\right\},$$

where $\Re e\left[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right]>0.$ (2.3.iii)

Remark 2.2.3: I have given only six new results in Examples 2.2.2' and 2.2.3' but it is possible to obtain several new results by using Examples 2.2.2 and 2.2.3 One needs to take different values of ν and a in these examples.

Example 2.2.4'.

(i) In Example 2.2.4 take $\nu = 1$ and then use Corollary 1, to obtain

$$\mathcal{L}_2^{-1}\left\{\frac{1}{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^3}\exp\left[-\frac{1}{\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2}\right];x,y\right\}$$
$$=-\frac{3}{4\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\left\{{}_1F_2\left[\begin{matrix}\frac{3}{2} & ; & \frac{xy}{x+y}\\ \frac{1}{2},1 & ; & \end{matrix}\right]-{}_1F_2\left[\begin{matrix}\frac{5}{2} & ; & \frac{xy}{x+y}\\ \frac{1}{2},1 & ; & \end{matrix}\right]\right\}$$

because

$${}_1F_2\left[\begin{matrix}\frac{3}{2} & ; & \frac{xy}{x+y}\\ \frac{1}{2},1 & ; & \end{matrix}\right]-{}_1F_2\left[\begin{matrix}\frac{5}{2} & ; & \frac{xy}{x+y}\\ \frac{1}{2},1 & ; & \end{matrix}\right]=-2\frac{xy}{x+y}\,{}_1F_2\left[\begin{matrix}\frac{5}{2} & ; & \frac{xy}{x+y}\\ \frac{3}{2},2 & ; & \end{matrix}\right]$$

we arrive at

$$\mathcal{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3}\exp\left[\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right];x,y\right\}=\frac{3xy}{2\pi^{\frac{1}{2}}(x+y)^{\frac{5}{2}}}\,{}_1F_2\left[\begin{matrix}\frac{5}{2}\ ;\\ \frac{3}{2},2\,;\end{matrix}\frac{xy}{x+y}\right],$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$ (2.4.i)

(ii) Now if we take v = -2 in Example 2.2.4 and then making use of Corollary 1, we arrive at

$$\mathcal{L}_2^{-1}\left\{\frac{3(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2+2}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3}\exp\left[\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right];x,y\right\}=\frac{3}{2\pi^{\frac{1}{2}}(x+y)^{\frac{3}{2}}}\,{}_1F_2\left[\begin{matrix}\frac{5}{2}\ ;\\ \frac{1}{2},1\,;\end{matrix}\frac{xy}{x+y}\right],$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$ (2.4.ii)

(iii) On substituting (2.4.i) into (2.4.ii) it follows that

$$\mathcal{L}_2^{-1}\left\{\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}\exp\left[\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right];x,y\right\}$$

$$=\frac{1}{2\pi^{\frac{1}{2}}(x+y)^{\frac{5}{2}}}\left\{(x+y)\,{}_1F_2\left[\begin{matrix}\frac{5}{2}\ ;\\ \frac{1}{2},1\,;\end{matrix}\frac{xy}{x+y}\right]-2xy\,{}_1F_2\left[\begin{matrix}\frac{5}{2}\ ;\\ \frac{3}{2},2\,;\end{matrix}\frac{xy}{x+y}\right]\right\},$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$ (2.4.iii)

Remark 2.2.4: The results established in Example 2.2.4' all are new results in two dimensions as well as the corresponding results in n-dimensions. Several other new results can be obtained from these examples and corollary 1, by taking different values of v and b.

CHAPTER 3. FURTHER NEW RESULTS ON N-DIMENSIONAL LAPLACE AND INVERSE LAPLACE TRANSFORMATIONS

3.1. Introduction

In Chapter 3 we established several new formulas for calculating Laplace transform and inverse Laplace transform pairs of N-dimensions from known one-dimensional Laplace transforms. The method we have employed for developing these results is similar to that used in Chapter 2.

This chapter consists of five theorems presented in four sections. We have given several examples on applications of these results in N and two dimensions.

3.2. The Image of Functions with the Argument $p_1(\overline{x^{-1}})$

In this section, first we prove two theorems and derive a corollary from Theorem 3.2.1. These results are used to compute the image of some special functions with the argument $p_1(\overline{x^{-1}})$ or $(x^{-1}+y^{-1})$ in N or two-dimensions respectively.

Theorem 3.2.1. Suppose that $f, x^{-\upsilon-1}\zeta(\frac{1}{x})$, and $\phi(2x^{\frac{1}{2}})$ are of class Ω.

Let

(i) $\mathcal{L}\{f(x);s\} = \phi(s)$,

(ii) $-\frac{d}{ds}\{s^{-\upsilon}\phi(\frac{1}{s})\} = \zeta(s)$,

(iii) $\mathcal{L}\{x^{-\upsilon-1}\zeta(\frac{1}{x});s\} = \eta(s)$,

(iv) $\mathcal{L}\{x^{-\frac{1}{2}}\phi(2x^{\frac{1}{2}});s\} = F(s)$,

(v) $\mathcal{L}\{x^{\frac{1}{2}}\phi(2x^{\frac{1}{2}});s\} = G(s)$,

and let $\frac{d}{ds}\{-s^{-v}\phi(\tfrac{1}{s})\}$ exist for $\Re$e s > c_0, where c_0 is a constant.

Then

$$\mathcal{L}_n\left\{\frac{(v-1)F[\overline{p_1(x^{-1})}]+2\overline{p_1(x^{-1})}G[\overline{p_1(x^{-1})}]}{p_n(x^{\frac{1}{2}})};\bar{s}\right\}=\frac{\pi^{\frac{n}{2}}}{p_n(s^{\frac{1}{2}})}\eta[\overline{p_1(s^{\frac{1}{2}})}],\ \text{where } \Re\text{e}\,[\overline{p_1(s^{\frac{1}{2}})}]>c_1$$

for some constant c_1 and n = 2, 3, ..., N and provided the integral involved exists. The existence conditions are given in Brychkov et al. [11; ch. 2].

Proof: Applying the definition of Laplace transform to (i) yields

$$\phi(s)=\int_o^\infty \exp(-st)f(t)dt \ \text{ where } \Re\text{e s} > c_0,\ c_0 \text{ is a constant.} \tag{1.1}$$

From (1.1) and (ii), we obtain

$$\zeta(s)=-\frac{d}{ds}\{s^{-v}\phi(\tfrac{1}{s})\}=vs^{-v-1}\phi(\tfrac{1}{s})-s^{-v}\frac{d}{ds}\phi(\tfrac{1}{s}) \tag{1.2}$$

Now, replacing s by $\frac{1}{x}$ in (2.1) and then multiplying the resulting equation by x^{-v-1} and taking the Laplace transform in (o, ∞) we obtain

$$\eta(s)=v\mathcal{L}\{\phi(x);s\}+\mathcal{L}\{x\phi'(x);s\}$$

Using formula (30) on page (6) (for n = 1) in Roberts and Kaufman [87], we obtain

$$\eta(s)=(v-1)\mathcal{L}\{\phi(x);s\}-s\frac{d}{ds}\mathcal{L}\{\phi(x);s\}=(v-1)\int_o^\infty \exp(-sx)\phi(x)dx+s\int_o^\infty x\exp(-sx)\phi(x)dx \tag{1.3}$$

Replacing s by $[\overline{p_1(s^{\frac{1}{2}})}]$ and multiplying both sides of (1.3) by $\overline{p_n(s^{\frac{1}{2}})}$, we arrive at

$$\overline{p_n(s^{\frac{1}{2}})}\eta[\overline{p_1(s^{\frac{1}{2}})}]=(v-1)\int_o^\infty \overline{p_n(s^{\frac{1}{2}})}\exp(-x\overline{p_1(s^{\frac{1}{2}})})\phi(x)dx$$

$$+\overline{p_n(s^{\frac{1}{2}})}\,\overline{p_1(s^{\frac{1}{2}})}\int_o^\infty \exp(-x\overline{p_1(s^{\frac{1}{2}})})x\phi(x)dx. \tag{1.4}$$

Next, we use the following results in Ditkin and Prudnikov [43]

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}})\overset{\bullet}{\underset{\bullet}{=}}(\pi x_i^{-\frac{1}{2}})\exp(-\tfrac{a^2}{4x_i}),\; s_i\exp(-as_i^{\frac{1}{2}})\overset{\bullet}{\underset{\bullet}{=}}\frac{a}{2\pi^{\frac{1}{2}}}x_i^{-\frac{3}{2}}\exp(-\tfrac{a^2}{4x_i})\text{ for } i = 1, 2, \ldots, \text{n}.$$

Equation (1.4) can be rewritten as

$$\begin{aligned} p_n(\overline{s^{\frac{1}{2}}})\eta[p_1(\overline{s^{\frac{1}{2}}})]\overset{n}{\underset{n}{=}}&\frac{(\upsilon-1)}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_o^\infty \exp(-\tfrac{x^2}{4}p_1(\overline{x^{-1}}))\phi(x)dx \\ &+\frac{p_1(\overline{x^{-1}})}{2\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_o^\infty x^2\phi(x)\exp(-\tfrac{x^2}{4}p_1(\overline{x^{-1}}))dx \end{aligned} \tag{1.5}$$

Now we substitute $u=\frac{x^2}{4}$ for integrands in (1.5) to obtain

$$\begin{aligned} p_n(\overline{s^{\frac{1}{2}}})\eta[p_1(\overline{s^{\frac{1}{2}}})]\overset{n}{\underset{n}{=}}&\frac{1}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})}[(\upsilon-1)\int_o^\infty \exp(-up_1(\overline{x^{-1}}))u^{-\frac{1}{2}}\phi(2u^{\frac{1}{2}})du \\ &+2p_1(\overline{x^{-1}})\int_o^\infty \exp(-up_1(\overline{x^{-1}}))u^{\frac{1}{2}}\phi(2u^{\frac{1}{2}})du \end{aligned} \tag{1.6}$$

Now we use (iv) and (v) in (1.6) to obtain

$$p_n(\overline{s^{\frac{1}{2}}})\eta[p_n(\overline{s^{\frac{1}{2}}})]\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})}\{(\upsilon-1)F[p_1(\overline{x^{-1}})]+2p_1(\overline{x^{-1}})G[p_1(\overline{x^{-1}})]\}$$

Hence

$$\mathfrak{L}_{\text{n}}\left\{\frac{(\upsilon-1)F[p_1(\overline{x^{-1}})]+2p_1(\overline{x^{-1}})G[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}=\frac{\pi^{\frac{n}{2}}}{p_n(\overline{s^{\frac{1}{2}}})}\eta[p_1(\overline{s^{\frac{1}{2}}})], \tag{1.7}$$

where n = 2, 3, ..., N, $\Re\text{e}\,[p_1(\overline{s^{\frac{1}{2}}})]>c_1$.

3.2.1. Applications of Theorem 3.2.1

As few examples of the use of Theorem 3.2.1, we shall construct certain functions with n variables, and calculate their Laplace transforms.

Example 3.2.1. Suppose that $f(x) = J_0(x)$ in Theorem 3.2.1. Then

$$\phi(s) = \frac{1}{(s^2+1)^{\frac{1}{2}}}, \text{ where } \Re e\, s > 0,$$

$$\zeta(s) = \frac{\upsilon s^2 + (\upsilon-1)}{s^{\upsilon}(s^2+1)^{\frac{3}{2}}},$$

$$\begin{aligned}\eta(s) &= \mathcal{L}\left\{\frac{\upsilon + (\upsilon-1)x^2}{(x^2+1)^{\frac{3}{2}}}; s\right\} \\ &= \mathcal{L}\left\{\frac{(\upsilon-1)}{(x^2+1)^{\frac{1}{2}}}; s\right\} + \mathcal{L}\left\{\frac{1}{(x^2+1)^{\frac{3}{2}}}; s\right\} \\ &= \frac{(\upsilon-1)\pi}{2}[H_o(s) - Y_o(s)] - \frac{\pi}{2}s[H_{-1}(s) - Y_{-1}(s)], \text{ where } \Re e\, s > 0.\end{aligned}$$

Also,

$$F(s) = \frac{1}{2}\mathcal{L}\left\{\frac{1}{(x^2+\frac{1}{4}x)^{\frac{1}{2}}}; s\right\} = \frac{1}{2}\exp(\frac{s}{8})K_o(\frac{s}{8}), \ \Re e\, x > 0, \ \Re e\, s > 0,$$

$$G(s) = \frac{1}{2}\mathcal{L}\left\{\frac{x^{\frac{1}{2}}}{(x+\frac{1}{4})^{\frac{1}{2}}}; s\right\} = \frac{1}{16}\exp(\frac{s}{8})[K_1(\tfrac{s}{8}) - K_o(\tfrac{s}{8})]$$

Using the result in Theorem 3.2.1, leads to the result

$$\mathcal{L}_n\left\{[4(\upsilon-1) - p_1(\overline{x^{-1}})]\exp(\frac{p_1(\overline{x^{-1}})}{8})K_o(\frac{p_1(\overline{x^{-1}})}{8}) + p_1(\overline{x^{-1}})\exp(\frac{p_1(\overline{x^{-1}})}{8})K_1(\frac{p_1(\overline{x^{-1}})}{8}); \bar{s}\right\}$$

$$= \frac{4\pi^{\frac{n+2}{2}}}{p_n(\overline{s^{\frac{1}{2}}})}\left\{(\upsilon-1)[H_o(p_1(\overline{s^{\frac{1}{2}}})) - Y_o(p_1(\overline{s^{\frac{1}{2}}}))] - p_1(\overline{s^{\frac{1}{2}}})[H_{-1}(p_1(\overline{s^{\frac{1}{2}}})) - Y_{-1}(p_1(\overline{s^{\frac{1}{2}}}))]\right\}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$, n = 2, 3, ..., N. (2.1)

Remark 3.2.1. Let v = 1 and n = 2 in (2.1) we obtain

$$\mathcal{L}_2\left\{\frac{x+y}{8xy}\exp\left(\frac{x+y}{8xy}\right)\left[K_1\left(\frac{x+y}{8xy}\right) - K_o\left(\frac{x+y}{8xy}\right)\right]; s_1, s_2\right\}$$

$$=\frac{\pi^2}{2}\left\{\frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}{s_1^{\frac{1}{2}}s_2^{\frac{1}{2}}}\left[Y_{-1}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})-H_{-1}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})\right]\right\}, \tag{2.1'}$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

We derive the following corollary from Theorem 3.2.1, by choosing $\nu = 1$.

Corollary 3.2.1. Assume the hypotheses of Theorem 3.2.1 for $\nu = 1$. Then

$$\mathfrak{L}_n\left\{\frac{p_1(\overline{x^{-1}})}{p_n(\overline{x^{\frac{1}{2}}})}G[p_1(\overline{x^{-1}})];\bar{s}\right\}=\frac{\pi^{\frac{n}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})}\eta[p_1(\overline{s^{\frac{1}{2}}})].$$

Theorem 3.2.2. Suppose f and $x^{-\frac{3}{2}}\phi(\frac{1}{x})$ belong to class Ω, where ϕ is the one-dimensional Laplace transform of f.

Let

(*i*) $\mathfrak{L}\left\{x^{\frac{-j^2+5j-2}{4}}f(x);s\right\}=F_j(s)$ for j = 0, 1, 2, and let $\frac{d}{ds}\left\{s^{-\upsilon}\zeta(\frac{1}{s^2})\right\}$ exist for $\Re e$ s > c_0, where c_0 is a constant.

(a) If $x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})$ belongs to $L_1[(0,\infty)\times(0,\infty)]$ and the following conditions hold:

(a1) $\mathfrak{L}\left\{x^{-\frac{3}{2}}\phi(\frac{1}{x});s\right\}=\zeta(s)$,

(a2) $-\frac{d}{ds}\left\{s^{-\upsilon}\zeta(\frac{1}{s^2})\right\}=\eta(s)$.

Then

$$\mathfrak{L}_n\left\{\frac{\upsilon F_o[p_1(\overline{x^{-1}})]-2p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}=\frac{\pi^{\frac{n-1}{2}}}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+1}}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}],$$

where $\Re e\,[\overline{p_1(s^{\frac{1}{2}})}] > c$, c is a constant. It is assumed that the integrals involved exist for n=2,3,...,N.

(b) Assume the condition (i), and replace (a1) and (a2) by

(b1) $L\left\{x^{-\frac{1}{2}}\phi(\tfrac{1}{x});s\right\} = \gamma(s)$,

(b2) $-\frac{d}{ds}\left\{s^{-v}\gamma(\tfrac{1}{s^2})\right\} = \varsigma(s)$,

and suppose that $\frac{d}{ds}\{s^{-v}\gamma(\frac{1}{s^2})\}$ exist for $\Re e\, s > c_1$, where c_1 is a constant.

Then

$$\mathcal{L}_n\left\{\frac{(v-1)\phi[\overline{p_1(x^{-1})}] - 2\overline{p_1(x^{-1})}F_2[\overline{p_1(x^{-1})}]}{p_n(x^{\frac{1}{2}})};\bar{s}\right\} = \frac{\pi^{\frac{n-1}{2}}}{p_n(s^{\frac{1}{2}})[\overline{p_1(s^{\frac{1}{2}})}]^v}\varsigma[(\overline{p_1(s^{\frac{1}{2}})})^{-1}],$$

where $\Re e\,[\overline{p_1(s^{\frac{1}{2}})}] > d$, a constant, n = 2, 3, ..., N. It is assumed that the integral on the left exists.

Proof (a): By the hypothesis and (a1) we get

$\zeta(s) = \int_0^\infty x^{-\frac{3}{2}}\phi(\frac{1}{x})\exp(-sx)dx = \int_0^\infty x^{-\frac{3}{2}}\exp(-sx)\left[\int_0^\infty f(u)\exp(-\frac{u}{x})du\right]dx$, where $\Re e\, s > c_0$ for some constant c_0 which leads to

$$\zeta(s) = \int_0^\infty\left[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{u}{x})f(u)du\right]dx \qquad (2.1)$$

Next we wish to interchange the order of the integral on the right side of (2.1). The integrand $x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})f(u) \in L_1[(0,\infty)\times(0,\infty)]$, according to Fubini's Theorem this process is permissible.

Therefore

$$\zeta(s)=\int_o^\infty f(u)\left[\int_o^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{u}{x})dx\right]du,\ \text{where}\ \Re e\ s>c_0. \tag{2.2}$$

A result in Roberts and Kaufman [87] regarding the inner integral in (2.2) is used to evaluate this integral as

$$\int_o^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{u}{x})dx=\pi^{\frac{1}{2}}u^{-\frac{1}{2}}\exp(-2u^{\frac{1}{2}}s^{\frac{1}{2}}).$$

Hence, (2.2) can be rewritten as

$$\zeta(s)=\pi^{\frac{1}{2}}\int_o^\infty u^{-\frac{1}{2}}f(u)\exp(-2u^{\frac{1}{2}}s^{\frac{1}{2}})du \tag{2.3}$$

Equation (2.3) together with (a2) shows that

$$\eta(s)=-\frac{d}{ds}\left\{s^{-\upsilon}\pi^{\frac{1}{2}}\int_o^\infty u^{-\frac{1}{2}}f(u)\exp(\tfrac{-2u^{\frac{1}{2}}}{s})du\right\}$$

$$=\pi^{\frac{1}{2}}s^{-\upsilon-1}\left[\upsilon\int_o^\infty u^{-\frac{1}{2}}f(u)\exp(-\tfrac{2u^{\frac{1}{2}}}{s})du-2s^{-1}\int_0^\infty f(u)\exp(-\tfrac{2u^{\frac{1}{2}}}{s})du\right],$$

so that

$$s^{\upsilon+1}\eta(s)=\pi^{\frac{1}{2}}\left[\upsilon\int_o^\infty u^{-\frac{1}{2}}f(u)\exp(-\tfrac{2u^{\frac{1}{2}}}{s})du-2s^{-1}\int_o^\infty f(u)\exp(-\tfrac{2u^{\frac{1}{2}}}{s})du\right], \tag{2.4}$$

where $\Re e\ s > c_0$.

Next, we replace s by $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and then multiply both sides of (2.4) by $p_n(\overline{s^{\frac{1}{2}}})$, to get

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\upsilon-1}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}]=\pi^{\frac{1}{2}}\upsilon\int_o^\infty u^{-\frac{1}{2}}p_n(\overline{s^{\frac{1}{2}}})f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})]du$$

$$-2\pi^{\frac{1}{2}}\int_o^\infty p_n(\overline{s^{\frac{1}{2}}})p_1(\overline{s^{\frac{1}{2}}})f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})]du \tag{2.5}$$

Now, we use the following operational relations given in Ditkin and Prudnikov [43]:

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}}) \doteq (\pi x_i)^{-\frac{1}{2}}\exp(-\tfrac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, n,$$

$$s_i\exp(-as_i^{\frac{1}{2}}) \doteq \frac{a}{2}\pi^{-\frac{1}{2}}x_i^{-\frac{3}{2}}\exp(-\tfrac{a^2}{4x_i}) \text{ for } i = 1, 2, \ldots, n.$$

Equation (2.5) can be rewritten as

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\upsilon-1}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}] \overset{n}{=} \frac{\upsilon}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_0^\infty u^{-\frac{1}{2}}f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du$$

$$-\frac{2p_1(\overline{x^{-1}})}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\int_0^\infty u^{\frac{1}{2}}f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du. \tag{2.6}$$

Equation (2.6) with (i) for $j = 0, 1$, yields to

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\upsilon-1}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}] \overset{n}{=} \frac{1}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\left\{\upsilon F_o[p_1(\overline{x^{-1}})] - 2(p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})]\right\}.$$

Thus

$$\mathcal{L}_n\left\{\frac{\upsilon F_o[p_1(\overline{x^{-1}})]-2p_1(\overline{x^{-1}})F_1[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\} = \frac{\pi^{\frac{n-1}{2}}}{p_n(\overline{s^{\frac{1}{2}}})[p_n(\overline{s^{\frac{1}{2}}})]^{\upsilon+1}}\eta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}] \tag{2.7}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > c$ and n = 2, 3, ..., N.

Proof (b): A similar method to part (a) can be used to prove part (b). The outline of the proof is as follows.

Making use of the hypothesis and (b1) yields

$$\gamma(s) = \pi^{\frac{1}{2}} \int_o^\infty \left[\int_o^\infty x^{-\frac{1}{2}} \exp(-sx - \tfrac{u}{x}) f(u) du \right] dx, \text{ where } \Re e\, s > c_1. \tag{2.8}$$

Using Fubini's Theorem to interchange the order of the integral on the right side of (2.8) and applying a result in Roberts and Kaufman [87] we obtain

$$\gamma(s) = \pi^{\frac{1}{2}} \int_o^\infty f(u) s^{-\frac{1}{2}} \exp(-2u^{\frac{1}{2}} s^{\frac{1}{2}}) du \tag{2.9}$$

Taking into account the condition (b2) we see that the equation (2.9) implies that

$$s^{v} \varsigma(s) = \pi^{\frac{1}{2}} (v-1) \int_o^\infty f(u) \exp(-\tfrac{2u^{\frac{1}{2}}}{s}) du - 2\pi^{\frac{1}{2}} s^{-1} \int_o^\infty u^{\frac{1}{2}} f(u) \exp(\tfrac{-2u^{\frac{1}{2}}}{s}) du, \text{ where } \Re e\, s > c_1. \tag{2.10}$$

Now, replacing s by $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and then multiplying both sides of (2.10) by $p_n(\overline{s^{\frac{1}{2}}})$ and then making use of the same two operational relations in part (a), equation (2.10) reads

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-v} \varsigma[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}] \overset{n}{\underset{n}{=}} \frac{1}{\pi^{\frac{n-1}{2}} p_n(\overline{x^{\frac{1}{2}}})} [(v-1) \int_o^\infty f(u) \exp[-up_1(\overline{x^{-1}})] dx$$
$$-2p_1(\overline{x^{-1}}) \int_o^\infty u f(u) \exp[-up_1(\overline{x^{-1}})] du. \tag{2.11}$$

Equation (2.11) with (i) for $i = 2$ and the definition of one-dimensional Laplace transform, leads to

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-v} \varsigma[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}]$$
$$\overset{n}{\underset{n}{=}} \frac{1}{\pi^{\frac{n-1}{2}} p_n(\overline{x^{\frac{1}{2}}})} \left\{ (v-1)\phi[p_1(\overline{x^{-1}})] - 2p_1(\overline{x^{-1}}) F_2[p_1(\overline{x^{-1}})] \right\}$$

Therefore,

$$\mathcal{L}_n\left\{\frac{(\upsilon-1)\phi[p_1(\overline{x^{-1}})]-2p_1(\overline{x^{-1}})F_2[p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-1}{2}}}{p_n(\overline{s^{\frac{1}{2}}})[p_n(\overline{s^{\frac{1}{2}}})]^{\upsilon}}\varsigma[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}], \tag{1.12}$$

where $\Re\text{e}\,[p_1(\overline{s^{\frac{1}{2}}})]>d$ for some constant d, n = 2, 3, ..., N.

3.2.2. Laplace Transforms of some Elementary and Special Functions with n Variables

The following examples will illustrate the applications of Theorem 3.2.2. We shall consider the function f to be an elementary or some special function to construct certain functions with n variables, and we calculate their Laplace transforms using Theorem 3.2.2. We will use the two dimensional case of these examples in Chapter 4 to solve certain types of partial differential equations.

Example 3.2.2. Let $f(x)=J_o(2x^{\frac{1}{2}})$. Then

$$\phi(s)=\frac{1}{s}\exp(-\frac{1}{s}),\ \Re\text{e}\, s>0,$$

$$F_j(s)=\frac{\Gamma(\frac{-j^2+5j+2}{4})}{s^{\frac{-j^2+5j+2}{4}}}\exp(-\frac{1}{s})\,{}_1F_1\left[\begin{matrix}\frac{j^2-5j+2}{4};\frac{1}{s}\\ 1\quad ;s\end{matrix}\right] \text{ for } j=0,1,2,\ \Re\text{e}\, s>0.$$

Now

$$\zeta(s)=\frac{\pi^{\frac{1}{2}}}{(s+1)^{\frac{1}{2}}},\ \Re\text{e}\, s>0,$$

so that

$$\eta(s)=\frac{\pi^{\frac{1}{2}}(\upsilon s^{2}+\upsilon-1)}{s^{\upsilon}(1+s^{2})^{\frac{3}{2}}},\ \Re\text{e}\, s>0.$$

Hence,

$$\mathfrak{L}_n\left\{\frac{\pi^{\frac{1}{2}}\upsilon\exp\left[-\frac{1}{p_1(\overline{x^{-1}})}\right]\ {}_1F_1\left[\begin{matrix}\frac{1}{2};\\ 1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]-\frac{2\pi^{\frac{1}{2}}}{2}\exp\left[-\frac{1}{p_1(\overline{x^{-1}})}\right]\ {}_1F_1\left[\begin{matrix}-\frac{1}{2};\\ 1\ ;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]}{[p_1(\overline{x^{-1}})]^{\frac{1}{2}}\,p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n-1}{2}}\cdot\pi^{\frac{1}{2}}\left[\upsilon[p_1(\overline{s^{\frac{1}{2}}})]^{-2}+\upsilon-1\right][p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon}}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\upsilon+1}\left[1+(p_1(\overline{s^{\frac{1}{2}}}))^{-2}\right]^{\frac{3}{2}}}.$$

Thus

$$\mathfrak{L}_n\left\{\frac{\exp\left[-\frac{1}{p_1(\overline{x^{-1}})}\right]\left\{\upsilon\ {}_1F_1\left[\begin{matrix}\frac{1}{2};\\ 1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]-{}_1F_1\left[\begin{matrix}-\frac{1}{2};\\ 1\ ;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]\right.}{p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{1}{2}}};\bar{s}\right\}=\frac{\pi^{\frac{n-1}{2}}\left[\upsilon+(\upsilon-1)p_1^{2}(\overline{s^{\frac{1}{2}}})\right]}{p_n(\overline{s^{\frac{1}{2}}})\left[1+p_1^{2}(\overline{s^{\frac{1}{2}}})\right]^{\frac{3}{2}}},\quad (2.1)$$

where $\Re\text{e}\ p_n(\overline{s^{\frac{1}{2}}})>0$.

Next, with the help of part (b), we have

$$\gamma(s)=\frac{\pi^{\frac{1}{2}}}{2(s+1)^{\frac{3}{2}}},\ \Re\text{e}\, s>1$$

Thus

$$\varsigma(s)=\frac{\pi^{\frac{1}{2}}(\upsilon s^{2}+\upsilon-3)}{2s^{\upsilon-2}(1+s^{2})^{\frac{5}{2}}},\ \Re\text{e}\, s>1.$$

Therefore

$$\mathfrak{L}_n\left\{\frac{1}{p_1(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\exp\left(-\frac{1}{p_1(\overline{x^{-1}})}\right)\left\{(\upsilon-1)-2\ {}_1F_1\left[\begin{matrix}-1\\ 1\end{matrix};\frac{1}{p_1(\overline{x^{-1}})}\right]\right\};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n}{2}}p_1(\overline{s^{\frac{1}{2}}})\left[\upsilon+(\upsilon-3)p_1^{2}(\overline{s^{\frac{1}{2}}})\right]}{2p_n(\overline{s^{\frac{1}{2}}})\left[1+p_1^{2}(\overline{s^{\frac{1}{2}}})\right]^{\frac{5}{2}}},\ \text{where } \Re\text{e}\,[p_1(\overline{s^{\frac{1}{2}}})]>1.\quad (2.2)$$

Remark 3.2.2: Equation (2.1) in the two-dimensional case for $\nu = 0$ or $\nu = 1$ reduces to the following new results, respectively.

$$\text{(i) } \mathfrak{L}_2\left\{\frac{\exp[-\frac{xy}{x+y}]\ {}_1F_1\left[\begin{matrix}-\frac{1}{2};\frac{xy}{x+y}\\ 1\ ;\end{matrix}\right]}{(x+y)^{\frac{1}{2}}};s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}}, \tag{2.1'}$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

$$\text{(ii) } \mathfrak{L}_2\left\{\frac{\exp[-\frac{xy}{x+y}]\left\{{}_1F_1\left[\begin{matrix}\frac{1}{2};\frac{xy}{x+y}\\ 1;\end{matrix}\right]-{}_1F_1\left[\begin{matrix}-\frac{1}{2};\frac{xy}{x+y}\\ 1\ ;\end{matrix}\right]\right\}}{(x+y)^{\frac{1}{2}}};s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}}$$

using the following result

$${}_1F_1\left[\begin{matrix}\frac{1}{2};\frac{xy}{x+y}\\ 1;\end{matrix}\right]-{}_1F_1\left[\begin{matrix}-\frac{1}{2};\frac{xy}{x+y}\\ 1\ ;\end{matrix}\right]=\frac{xy}{x+y}\,{}_1F_1\left[\begin{matrix}\frac{1}{2};\frac{xy}{x+y}\\ 2;\end{matrix}\right].$$

The last formula can be simplified as

$$\mathfrak{L}_2\left\{\exp[-\tfrac{xy}{x+y}]xy(x+y)^{\frac{1}{2}}\ {}_1F_1\left[\begin{matrix}\frac{1}{2};\frac{xy}{x+y}\\ 2;\end{matrix}\right];s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}} \tag{2.1''}$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Using the following version of F_j

$$F_j(s)=\frac{\Gamma(\frac{-j^2+5j+2}{4})}{s^{\frac{-j^2+5j+2}{4}}}\ {}_1F_1\left[\begin{matrix}\frac{-j^2+5j+2}{4};-\frac{1}{s}\\ 1\ ;\end{matrix}\right] \text{ for } j=0,1,2.$$

The Formula (2.1) turns out to be

$$\mathfrak{L}_n\left\{\frac{v\,{}_1F_1\left[\begin{matrix}\frac{1}{2};-\frac{1}{p_1}\\ 1;\end{matrix}\right]-{}_1F_1\left[\begin{matrix}\frac{3}{2};-\frac{1}{p_1(\overline{x^{-1}})}\\ 1;\end{matrix}\right]}{p_1(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{1}{2}}};\bar{s}\right\}=\frac{\pi^{\frac{n-1}{2}}[v+(v-1)p_1^2(\overline{s^{-\frac{1}{2}}})]}{p_n(\overline{s^{\frac{1}{2}}})[1+p_1^2(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}}, \tag{2.3}$$

where $\Re e\,[p_1(\overline{s^{\frac{1}{2}}})] > 0$.

Special cases of (2.3) for n = 2, ν = 0 or ν = 1 turn out to be

$$L_2\left\{\frac{1}{(x+y)^{\frac{1}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2};\\ 1;\end{matrix}-\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \tag{2.3'}$$

$$L_2\left\{xy(x+y)^{\frac{1}{2}}\,{}_1F_1\left[\begin{matrix}\frac{1}{2};\\ 2;\end{matrix}-\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \tag{2.3''}$$

Remark 3.2.3: If we let n = 2 and ν = 1 or ν = 3, from the equation (2.2) we deduce the following results, respectively

(i)
$$L_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right)\,{}_1F_1\left[\begin{matrix}-1;\\ 1\ ;\end{matrix}\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{\pi(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})[2(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2-1]}{4(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{5}{2}}}, \tag{2.2'}$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1$.

(ii)
$$L_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right)\left\{1-{}_1F_1\left[\begin{matrix}-1;\\ 1\ ;\end{matrix}\frac{xy}{x+y}\right]\right\};s_1,s_2\right\}=\frac{3\pi(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{2(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{5}{2}}}, \tag{2.2''}$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1$.

Notice, with the help of (2.2') from (2.2'') we arrive at the following result

$$L_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi}{2}\,\frac{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}}. \tag{2.2'''}$$

This is the same as the result (2.109) in Ditkin and Prudnikov [43; p. 140].

Furthermore, with the help of (2.2''') and the operational relation (47) in Voelker and Doetsch [107; p. 159], we derive the following new results.

$$\mathfrak{L}_2\left\{\left(\frac{y}{x}\right)^{\frac{1}{2}}\cdot\frac{1}{x+y}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi}{s_2^{\frac{1}{2}}\left[1+\left(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}\right)^2\right]^{\frac{1}{2}}},\ \Re\mathrm{e}\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1. \tag{2.2^{IV}}$$

$$\mathfrak{L}_2\left\{\left(\frac{x}{y}\right)^{\frac{1}{2}}\cdot\frac{1}{x+y}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi}{s_1^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{1}{2}}}, \tag{2.2^{V}}$$

where $\Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1$.

Example 3.2.3. If we let $f(x)$ to be ${}_0F_1\left[\begin{matrix}\cdot\,;\\1;\end{matrix}x\right]$, $x^{\frac{1}{4}}J_{\frac{1}{2}}(2x^{\frac{1}{2}})$ or $\cos 2x^{\frac{1}{2}}$ then using Theorem 3.2.2 we derive the following results

(a) *(i)* $$\mathfrak{L}_n\left\{\frac{1}{[p_1(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\left\{\nu\,{}_1F_1\left[\begin{matrix}\frac{1}{2};\\1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]-{}_1F_1\left[\begin{matrix}\frac{3}{2};\\1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]\right\};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n-1}{2}}}{p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})}\left\{(\nu-1)\,{}_2F_1\left[\begin{matrix}1,\frac{1}{2};\\1;\end{matrix}\frac{1}{p_1^2(\overline{s^{\frac{1}{2}}})}\right]-\frac{1}{p_1(\overline{s^{\frac{1}{2}}})}\,{}_2F_1\left[\begin{matrix}2,\frac{3}{2};\\2\ ;\end{matrix}\frac{1}{p_1^2(\overline{s^{\frac{1}{2}}})}\right]\right\} \tag{3.1}$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>1$ and $n=2,3,\ldots,N$.

(b) *(i)* $$\mathfrak{L}_n\left\{\frac{1}{p_1(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\left\{(\nu-1)\,{}_1F_1\left[\begin{matrix}1;\\1;\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]-\frac{3\pi^{\frac{1}{2}}}{2p_1^{\frac{1}{2}}(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\,{}_1F_1\left[\begin{matrix}\frac{5}{2};\\\frac{1}{2};\end{matrix}\frac{1}{p_1(\overline{x^{-1}})}\right]\right\};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})p(\overline{s^{\frac{1}{2}}})}\left\{(\nu-3)\,{}_2F_1\left[\begin{matrix}1,\frac{3}{2};\\1\ ;\end{matrix}\frac{1}{p_1^2(\overline{x^{-1}})}\right]-\frac{3}{p_1^2(\overline{x^{\frac{1}{2}}})}\,{}_2F_1\left[\begin{matrix}2,\frac{3}{2};\\2\ ;\end{matrix}\frac{1}{p_1^2(\overline{x^{-1}})}\right]\right\}, \tag{3.2}$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>1,\ n=1,2,\ldots,N$.

(a) *(ii)* $$\mathfrak{L}_n\left\{\frac{1}{p_1(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\left\{\nu\,{}_1F_1\left[\begin{matrix}1\ ;\\\frac{3}{2};\end{matrix}-\frac{1}{p_1(\overline{x^{-1}})}\right]-2\,{}_1F_1\left[\begin{matrix}2;\\\frac{3}{2};\end{matrix}-\frac{1}{p_1(\overline{x^{-1}})}\right]\right\};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n}{2}}\left[\nu+(\nu-2)p_1^2(\overline{s^{\frac{1}{2}}})\right]}{2p_n(\overline{s^{\frac{1}{2}}})\left[1+p_1^2(\overline{s^{\frac{1}{2}}})\right]^2},\quad \Re e[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=2,3,\ldots,N. \tag{3.3}$$

(b) *(iii)* $\mathfrak{L}_n\left\{\frac{(\nu-1)}{p_1^{\frac{3}{2}}(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\exp\left(-\frac{1}{p_1(\overline{x^{-1}})}\right)-\frac{8\pi^{-\frac{1}{2}}}{p_1^2(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}{}_1F_1\left[\begin{matrix}3;\\ \frac{3}{2};\end{matrix}-\frac{1}{p_1(\overline{x^{-1}})}\right];\bar{s}\right\}$

$$=\frac{\pi^{\frac{n-1}{2}}p_1^3(\overline{s^{\frac{1}{2}}})\left[(\nu-2)+(\nu+2)p_1^2(\overline{s^{\frac{1}{2}}})\right]}{p_n(\overline{s^{\frac{1}{2}}})\left[1+p_1^2(\overline{s^{\frac{1}{2}}})\right]^3},\quad \Re e\,[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=1,2,\ldots,N. \tag{3.4}$$

(a) *(iii)* $\mathfrak{L}_n\left\{\frac{1}{p_1^{\frac{3}{2}}(\overline{x^{-1}})p_n(\overline{x^{\frac{1}{2}}})}\left[2+(\nu-1)p_1(\overline{x^{-1}})\exp\left[-\frac{1}{p_1(\overline{x^{-1}})}\right]\right];\bar{s}\right\}$

$$=\frac{\pi^{\frac{n-1}{2}}p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\left[\frac{(\nu+1)+(\nu-1)p_1^2(\overline{s^{\frac{1}{2}}})}{\left(1+p_1^2(\overline{s^{\frac{1}{2}}})\right)^2}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=2,3,\ldots,N. \tag{3.5}$$

Example 3.2.3'. (Two-dimensions)

(i) Upon substituting n=2 and $\nu=1$ in (3.1) we arrive at a new result as follows

$$\mathfrak{L}_2\left\{\frac{(xy)}{(x+y)^{\frac{3}{2}}}{}_1F_1\left[\begin{matrix}\frac{5}{2};\\ 2;\end{matrix}\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3}{}_2F_1\left[\begin{matrix}2,\frac{3}{2};\\ 2\ ;\end{matrix}\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}\right], \tag{3.1'}$$

where $\Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>1$.

(ii) On substitution n=2 and $\nu=2$ or $\nu=0$ in (3.3), we obtain the following results, respectively

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{3}{2}}}{(x+y)^2}{}_1F_1\left[\begin{matrix}2; \\ \frac{5}{2};\end{matrix} -\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{3\pi}{4(s_1s_2)^{\frac{1}{2}}\left[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\right]^2},\ \Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (3.3')$$

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{1}{2}}}{(x+y)}{}_1F_1\left[\begin{matrix}2; \\ \frac{3}{2};\end{matrix} -\frac{xy}{x+y}\right];s_1,s_2\right\}=\frac{\pi(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2}{2(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]^2},\ \Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (3.3'')$$

(iii) If we let n=2, $\nu=1$ in (3.5), we obtain

$$\mathcal{L}_2\left\{\frac{xy}{(x+y)^{\frac{3}{2}}}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^2},\ \Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (3.5')$$

Furthermore if we substitute $\nu=3$ and with the help of (3.5'), we arrive at

$$\mathcal{L}_2\left\{\frac{1}{x+y}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{\pi^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]},\Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (3.5'')$$

The operational relations (3.5') and (3.5") both are well-known results.

Similarly many more double Laplace transforms can be derived by taking different values of ν in (3.1), (3.2), (3.3), (3.4) and (3.5).

Example 3.2.4. Suppose $f(x)=x^{\alpha}\exp(-bx)$. Then

$$\phi(s)=\frac{\Gamma(\alpha+1)}{(s+b)^{\alpha+1}},\ \Re e\,\alpha>-1,\ \Re e\,s>-\Re e\,b,$$

$$\xi(s)=(\tfrac{2}{b})^{\alpha+1}\Gamma(\alpha+1)\Gamma(\alpha+\tfrac{1}{2})\exp(\tfrac{s}{2b})D_{-2\alpha-1}[(\tfrac{2s}{b})],\ \Re e\,\alpha>-\tfrac{1}{2}$$

For $\nu=0$,

$$\eta(s)=-(\tfrac{2}{b})^{\alpha+\frac{3}{2}}\Gamma(\alpha+1)\Gamma(\alpha+\tfrac{1}{2})(2\alpha+1)\cdot\frac{\exp(\frac{1}{2bs^2})}{s^2}D_{-2\alpha-2}[(\tfrac{2}{b})^{\frac{1}{2}}\tfrac{1}{s}],$$

where $\Re e\,\alpha > -\frac{1}{2}$, $\Re e\, s > 0$ and $|\arg \frac{1}{b}| < \pi$.

Next,

$$F_1(s) = \frac{\Gamma(\alpha+\frac{3}{2})}{(s+b)^{\alpha+\frac{3}{2}}}, \quad \Re e\, s > -\Re e\, b,\ \Re e\,\alpha > -\frac{3}{2}.$$

Therefore, from (2.7) for $\nu = 0$, we obtain

$$\mathfrak{L}_n\left\{\frac{p_1(\overline{x^{-1}})}{p_n(\overline{x^{\frac{1}{2}}})[b+p_1(\overline{x^{-1}})]^{\alpha+\frac{3}{2}}};\bar{s}\right\}$$

$$= \pi^{\frac{n-1}{2}}(\tfrac{2}{b})^{\alpha+\frac{3}{2}}\frac{\Gamma(\alpha+1)p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\exp[\tfrac{1}{2b}p_1{}^2(\overline{s^{\frac{1}{2}}})]D_{-2\alpha-2}[(\tfrac{2}{b})^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})], \tag{4.1}$$

where $\Re e\,\alpha > -\frac{1}{2}$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})] > 0$ and $n = 2,3,\ldots,N$.

Furthermore,

$$\gamma(s) = -(\tfrac{2}{b})^{\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+\tfrac{3}{2})s^{-\frac{1}{2}}\exp(\tfrac{s}{2b})D_{-2\alpha-2}[(\tfrac{2}{b})^{\frac{1}{2}}\tfrac{1}{s}],\ \Re e\,\alpha > -1,$$

for $\nu = 1$,

$$\zeta(s) = \frac{-(\frac{2}{b})^{\alpha+2}\Gamma(\alpha+1)\Gamma(\alpha+\frac{3}{2})(2\alpha+2)}{s^2}\exp(\tfrac{1}{2bs^2})D_{-2\alpha-3}[(\tfrac{2}{b})^{\frac{1}{2}}\tfrac{1}{s}],$$

where $\Re e\,\alpha > -1$, $\Re e\, s > 0$ and $|\arg \frac{1}{b}| < \pi$.

Also,

$$F_2(s) = \frac{\Gamma(\alpha+2)}{(s+b)^{\alpha+2}}, \quad \Re e\, s > -\Re e\, b,\ \Re e\,\alpha > -2.$$

Hence, from (2.12) for $\nu = 1$, we arrive at

$$\mathbf{L}_2\left\{\frac{p_1(\overline{x^{-1}})}{p_n(\overline{x^{\frac{1}{2}}})[b+p_1(\overline{x^{-1}})]^{\alpha+2}};\bar{s}\right\}$$

$$=\pi^{\frac{n-1}{2}}\left(\tfrac{2}{b}\right)^{\alpha+2}\cdot\frac{\Gamma(\alpha+\frac{3}{2})p_1(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\exp\left[\tfrac{1}{2b}p_1^{\,2}(\overline{s^{\frac{1}{2}}})\right]D_{-2\alpha-3}\left[\left(\tfrac{1}{2b}\right)^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})\right], \tag{4.2}$$

where $\Re e\,\alpha>-1$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0$ and $n=2,3,\ldots,N$.

Notice, (4.1) can be deduced from (4.2) by replacing α by $\alpha-\frac{1}{2}$. But this is not always true.

Remark 3.2.4: If we let n=2 in (4.1) we obtain the following new result

$$\mathbf{L}_2\left\{\frac{(x+y)(xy)^{\alpha}}{(x+y+bxy)^{\alpha+\frac{3}{2}}};s_1,s_2\right\}$$

$$=\frac{\left(\frac{2}{b}\right)^{\alpha+\frac{3}{2}}\pi^{\frac{1}{2}}\Gamma(\alpha+1)(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}}\exp\left[\tfrac{1}{2b}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2\right]D_{-2\alpha-2}\left[\left(\tfrac{2}{b}\right)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})\right]. \tag{4.1'}$$

One can develop many other operational relations from given functions f in Example (3.2.4) via using (2.7) or (2.12) for different values of ν.

3.3. The Original of Functions with the Argument $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$

In this section we will establish Theorem 3.3.1 in two parts. The proof of part (b) is similar to that employed for developing the result in part (a), so that we give the proof of part (a) in detail and state the result in part (b) without proof. Regarding their applications, we used these results to compute the original of a few special functions with the argument $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ or $(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{-1}$ in N or two-dimensions, respectively.

Theorem 3.3.1. Let f and $x^{-\frac{3}{2}}f(x)$ be of class Ω and let $\phi(x)$ be a one-dimensional Laplace transform of f.

Suppose

(i) $\mathcal{L}\left\{x^{\frac{3j^4-22j^3+45j^2+22j-72}{48}}f(x);s\right\}=G_j(s)$ for $j=0,1,2,3,4$.

(a) If

(a1) $\mathcal{L}\left\{x^{-\frac{5}{2}}\phi(\frac{1}{x});s\right\}=\xi(s)$,

(a2) $-\frac{d}{ds}\{s^{-\nu}\xi(\frac{1}{s^2})\}=\Theta(s)$.

Assuming $x^{-\frac{5}{2}}\phi(\frac{1}{x})$ is of class Ω and $\frac{d}{ds}\{s^{-\nu}\xi(\frac{1}{s^2})\}$ exists for $\Re e\, s>c_0$ for some fixed c_0. Furthermore, $x^{-\frac{5}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Then

$$\mathcal{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\Theta[(p_1(\overline{s^{\frac{1}{2}}}))^{-1}];\bar{x}\right\}$$

$$=\frac{1}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\left\{\frac{\nu}{2}G_0\left[p_1(\overline{x^{-1}})\right]+(\nu+1)p_1(\overline{x^{-1}})G_1\left[p_1(\overline{x^{-1}})\right]-2[p_1(\overline{x^{-1}})]^2G_2\left[p_1(\overline{x^{-1}})\right]\right\},$$

provided that the integral involved in the left-side exists for $n=2,3,\ldots,N$.

(b) Assume that the condition (i) holds and let $x^{-\frac{5}{2}}\phi(\frac{1}{x})$ be of class Ω, and replace (a1) and (a2) by the following

(b1) $\mathcal{L}\left\{x^{\frac{1}{2}}\phi(\frac{1}{x});s\right\}=\gamma(s)$.

(b2) $-\frac{d}{ds}\left\{s^{-\nu}\gamma\left(\frac{1}{s^2}\right)\right\}=\sigma(s)$.

Then

$$\mathfrak{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu-2}}\sigma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right];\bar{s}\right\}$$

$$=\frac{1}{\pi^{\frac{n-1}{2}}\overline{p_n(x^{\frac{1}{2}})}}\left\{\frac{\nu-3}{2}\phi[p_1(\overline{x^{-1}})]+(\nu-2)p_1(\overline{x^{-1}})G_3[p_1(\overline{x^{-1}})]-2[p_1(\overline{x^{-1}})]^2G_4[p_1(\overline{x^{-1}})]\right\},$$

where $n=2,3,\ldots,N$ and provided that $\frac{d}{ds}\{s^{-\nu}\gamma(\frac{1}{s^2})\}$ exists for $\Re e\, s>c_1$ for some fixed c_1. Moreover, $x^{\frac{1}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$ and the integral involved in the left-side exists. The existence conditions for two as well as n-dimensional inverse Laplace transformations are given in Brychkov et al. [11; ch. 2].

Proof: We know that

$$\phi(x)=\int_0^{\infty}\exp(-st)dt \text{ where } \Re e\, s>c_0, \text{ for some fixed } c_0. \tag{3.1}$$

From (3.1) and (a1), we obtain

$$\xi(s)=\int_0^{\infty}\left[\int_0^{\infty}x^{-\frac{5}{2}}\exp(-sx-\tfrac{u}{x})f(u)du\right]dx. \tag{3.2}$$

The integrand $x^{-\frac{5}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, so, by Fubini's Theorem, interchanging the order of the integration on the right-side of (3.2) is permissible. Hence

$$\xi(x)=\int_0^{\infty}f(u)[\int_0^{\infty}x^{-\frac{5}{2}}\exp(-sx-\tfrac{u}{x})dx]du,\ \Re e\, s>c_0. \tag{3.2'}$$

A result in Roberts and Kaufman[87] regarding the inner integral in (3.2') is used to evaluate this integral as

$$\xi(x)=\frac{\pi^{\frac{1}{2}}}{2}\int_0^{\infty}f(u)[u^{-\frac{3}{2}}+2u^{-1}s^{\frac{1}{2}}]\exp(-2u^{\frac{1}{2}}s^{\frac{1}{2}})du$$

A little calculus yields the result that

$$s^{\nu+1}\Theta(s)=\frac{\pi^{\frac{1}{2}}}{2}[\nu\int_0^\infty u^{-\frac{3}{2}}f(u)\exp(\frac{-2u^{\frac{1}{2}}}{s})du+2\nu s^{-1}\int_0^\infty u^{-1}f(u)\exp(\frac{-2u^{\frac{1}{2}}}{s})du$$

$$-4s^{-2}\int_0^\infty u^{-\frac{1}{2}}f(u)\exp(\frac{-2u^{\frac{1}{2}}}{s})du],\ \Re e\, s>c_0. \tag{3.3}$$

Replacing s by $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and multiplying both sides of (3.3) by $p_n(\overline{s^{\frac{1}{2}}})$, we arrive at

$$\begin{aligned}[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}p_n(\overline{s^{\frac{1}{2}}})\Theta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]&=\frac{\pi^{\frac{1}{2}}}{2}\left[\nu\int_0^\infty u^{-\frac{3}{2}}f(u)p_n(\overline{s^{\frac{1}{2}}})\exp[-2p_1(\overline{s^{\frac{1}{2}}})]du\right.\\ &\left.+2\nu\int_0^\infty f(u)p_1(\overline{s^{\frac{1}{2}}})p_n(\overline{s^{\frac{1}{2}}})\exp[-2u^{\frac{1}{2}}p_1(\overline{s^{\frac{1}{2}}})]du-4\int_0^\infty[p_1(\overline{s^{\frac{1}{2}}})]^2p_n(\overline{s^{\frac{1}{2}}})u^{-\frac{1}{2}}f(u)du\right]\end{aligned} \tag{3.4}$$

Next, using the following well-known operational relations

$$s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}})\overset{\bullet}{\underset{\bullet}{=}}\pi^{-\frac{1}{2}}x_i^{-\frac{1}{2}}\exp(-\frac{a^2}{4x_i}),$$

$$s_i\exp(-as_i^{\frac{1}{2}})\overset{\bullet}{\underset{\bullet}{=}}\frac{a}{2}\pi^{-\frac{1}{2}}x_i^{-\frac{3}{2}}\exp(-\frac{a^2}{4x_i}),$$

$$s_i^{\frac{3}{2}}\exp(-as_i^{\frac{1}{2}})\overset{\bullet}{\underset{\bullet}{=}}\frac{a^2-2x_i}{4}\pi^{-\frac{1}{2}}x_i^{-\frac{5}{2}}\exp(-\frac{a^2}{4x_i})\ \text{for } i=1,2,\dots,n.$$

into (3.4) yields the result that

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\Theta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]\overset{n}{\underset{n}{=}}\frac{\pi^{\frac{1}{2}}}{2}\left\{\nu\int_0^\infty\frac{1}{\pi^{\frac{n}{2}}p_n(\overline{s^{\frac{1}{2}}})}u^{-\frac{1}{2}}f(u)\exp[-up_1(\overline{x^{-1}})]du\right.$$

$$+2\nu\int_0^\infty\frac{1}{\pi^{\frac{n}{2}}p_n(\overline{x^{-\frac{1}{2}}})}u^{\frac{1}{2}}x_1^{-\frac{3}{2}}\exp(-\frac{u}{x_1})\exp[-up_1(\overline{x^{-1}})]u^{-1}f(u)du$$

$$+\dots+\int_0^\infty\frac{\cdot 1}{\pi^{\frac{n}{2}}p_n(\overline{x^{-\frac{1}{2}}})}u^{\frac{1}{2}}x_n^{-\frac{3}{2}}\exp(-\frac{u}{x_n})\exp\left[-up_1(\overline{x^{-1}})\right]u^{-1}f(u)du$$

$$-\frac{4}{\pi^{\frac{n}{2}}}\left\{\frac{1}{4}\int_0^\infty\left[\frac{(4u-2x_1)}{p_{n-1}(x_1^{\frac{1}{2}})}x_1^{-\frac{5}{2}}+\ldots+\frac{(4u-2x_n)}{p_{n-1}(x_n^{\frac{1}{2}})}x_n^{-\frac{5}{2}}\right]\exp[-u^{\frac{1}{2}}p_1(\overline{x^{-1}})]u^{-\frac{1}{2}}f(u)du\right.$$

$$+2\int_0^\infty\left[\frac{ux_1^{-\frac{3}{2}}x_2^{-\frac{3}{2}}}{p_{n-2}(x_{12}^{\frac{1}{2}})}+\ldots+\frac{ux_1^{-\frac{3}{2}}x_n^{-\frac{3}{2}}}{p_n(x_{1n}^{\frac{1}{2}})}\right]\exp[-u^{\frac{1}{2}}p_1(\overline{x^{-1}})]u^{-\frac{1}{2}}f(u)du$$

$$+2\int_0^\infty\left[\frac{ux_2^{-\frac{3}{2}}x_3^{-\frac{3}{2}}}{p_{n-2}(x^{\frac{1}{2}}{}_{23})}+\ldots+\frac{ux_2^{-\frac{3}{2}}x_n^{-\frac{3}{2}}}{p_{n-2}(x_{2n}^{\frac{1}{2}})}\right]\exp[-u^{\frac{1}{2}}p_1(\overline{x^{-1}})]u^{-\frac{1}{2}}f(u)du$$

$$\left.+\ldots+2\int_0^\infty\frac{ux_{n-1}^{-\frac{3}{2}}x_n^{-\frac{3}{2}}}{p_{n-2}(x_{n-1n}^{\frac{1}{2}})}\exp[-u^{\frac{1}{2}}p_1(\overline{x^{-1}})]u^{-\frac{1}{2}}f(u)du\right\}\Bigg\} \tag{3.5}$$

A little algebra yields the result that

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\Theta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]$$

$$\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\left\{\frac{\nu}{2}\int_0^\infty u^{-\frac{3}{2}}f(u)\exp[-up_1(\overline{x^{-1}})]du\right.$$

$$\left.+(\nu+1)p_1(\overline{x^{-1}})\int_0^\infty u^{-\frac{1}{2}}f(u)\exp[-up_1(\overline{x^{-1}})]du-2[p_1(\overline{x^{-1}})]^2\int_0^\infty u^{\frac{1}{2}}f(u)\exp[-up_1(\overline{x^{-1}})]du\right\} \tag{3.6}$$

Using (i) for $j=0,1,$ and 2, we obtain

$$\mathcal{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\Theta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right];\overline{x}\right\}$$

$$=\frac{1}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})}\left\{\frac{\nu}{2}G_0\left[p_1(\overline{x^{-1}})\right]+(\nu+1)p_1(\overline{x^{-1}})G_1\left[p_1(\overline{x^{-1}})\right]\right.$$

$$\left.-2[p_1(\overline{x^{-1}})]^2G_2\left[p_1(\overline{x^{-1}})\right]\right\}, \tag{3.7}$$

where $n=2,3,\ldots,N$.

3.3.1. Examples Based Upon Theorem 3.3.1

Example 3.3.1. Suppose that $f(x)=x^{\frac{1}{2}}J_1(2x^{\frac{1}{2}})$. Then

$$\phi(x)=\tfrac{1}{s^2}\exp(-\tfrac{1}{s}),$$

$$\xi(s)=\frac{\pi^{\frac{1}{2}}}{(s+1)^{\frac{1}{2}}},$$

$$\Theta(s)=\frac{\pi^{\frac{1}{2}}(\nu s^2+\nu-1)}{s^{\nu}(1+s^2)^{\frac{3}{2}}},$$

$$G_j(s)=\frac{\Gamma\left(\frac{3j^4-22j^3+45j^2-22j+24}{48}\right)}{s^{\left(\frac{3j^4-22j^3+45j^2-22j+24}{48}\right)}}\,{}_1F_1\left[\begin{matrix}\frac{3j^4-22j^3+45j^2-22j+24}{48}; & -\frac{1}{s}\\ 2 & ;\end{matrix}\right] \text{ for } j=0,1,2,3,4,$$

where $\Re e\, s>0$. A little algebra yields the result that

$$\mathfrak{L}_n^{-1}\left\{\frac{[\nu+(\nu-1)p_1{}^2(\overline{s^{\frac{1}{2}}})]}{p_n(\overline{s^{\frac{1}{2}}})[1+p_1{}^2(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}};\bar{x}\right\}=\frac{1}{\pi^{\frac{n-1}{2}}p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^{\frac{1}{2}}}$$

$$\cdot\left\{\frac{\nu}{2}\,{}_1F_1\left[\begin{matrix}\frac{1}{2}; & -\frac{1}{p_1(\overline{x^{-1}})}\\ 2; & \end{matrix}\right]+\frac{\nu+1}{2}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; & -\frac{1}{p_1(\overline{x^{-1}})}\\ 2; & \end{matrix}\right]-\frac{3}{2}\,{}_1F_1\left[\begin{matrix}\frac{5}{2}; & -\frac{1}{p_1(\overline{x^{-1}})}\\ 2; & \end{matrix}\right]\right\},$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=2,3,\ldots,N.$

Next, we use part (b) to find that

$$\gamma(s)=\frac{15\pi^{\frac{1}{2}}}{8(s+1)^{\frac{7}{2}}},\ \Re e\, s>0,$$

$$\sigma(s)=\frac{15\pi^{\frac{1}{2}}(\nu s^2+\nu-7)}{8s^{\nu-6}(1+s^2)^{\frac{9}{2}}},\ \Re e\, s>0.$$

Thus,

$$\mathcal{L}_n^{-1}\left\{\frac{p_1{}^3(\overline{s^{\frac{1}{2}}})[\nu+(\nu-7)p_1{}^2(\overline{s^{\frac{1}{2}}})]}{p_n(\overline{s^{\frac{1}{2}}})[1+p_1{}^2(\overline{s^{\frac{1}{2}}})]^{\frac{9}{2}}};\bar{x}\right\}$$

$$=\frac{8}{15\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})p_1(\overline{x^{-1}})}\left\{\frac{\nu-3}{2}\exp\left[-\frac{1}{p_1(\overline{x^{-1}})}\right]+2(\nu-2)\,{}_1F_1\left[\begin{matrix}3;\\2;\end{matrix}-\frac{1}{p_1(\overline{x^{-1}})}\right]-12\,{}_1F_1\left[\begin{matrix}4;\\2;\end{matrix}-\frac{1}{p_1(\overline{x^{-1}})}\right]\right\}, \quad (3.1)$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=2,3,\ldots,N.$

Example 3.3.1'. (Two-dimensions)

If we let $n=2$ and $\nu=-4$ in (3.1), we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^3[4+11(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2]}{(s_1s_2)^{\frac{1}{2}}[1+(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2]^{\frac{9}{2}}};x,y\right\}$$

$$=\frac{8(xy)^{\frac{3}{2}}}{15\pi(x+y)^2}\left\{\frac{7}{2}\exp\left(-\frac{xy}{x+y}\right)+12\left\{{}_1F_1\left[\begin{matrix}3;\\2;\end{matrix}-\frac{xy}{x+y}\right]-{}_1F_1\left[\begin{matrix}4;\\2;\end{matrix}-\frac{xy}{x+y}\right]\right\}\right\}.$$

Using the following results,

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{3}{2}}}{(x+y)^2}\exp\left(-\frac{xy}{x+y}\right);s_1,s_2\right\}=\frac{3\pi(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})}{4(s_1s_2)^{\frac{1}{2}}[1+(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2]^{\frac{5}{2}}},\quad \Re e[s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}}]>0.$$

and

$${}_1F_1\left[\begin{matrix}\frac{3}{2};\\2;\end{matrix}-\frac{xy}{x+y}\right]-{}_1F_1\left[\begin{matrix}4;\\2;\end{matrix}-\frac{xy}{x+y}\right]=\frac{xy}{x+y}\,{}_1F_1\left[\begin{matrix}4;\\3;\end{matrix}-\frac{xy}{x+y}\right],$$

we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{6(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^3[1+8(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2]-7(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}[1+(s^{\frac{1}{2}}{}_1+s_2{}^{\frac{1}{2}})^2]^{\frac{9}{2}}};x,y\right\}$$

$$=\frac{32}{\pi}\frac{(xy)^{\frac{5}{2}}}{(x+y)^3}\,{}_1F_1\left[\begin{matrix}4;\\3;\end{matrix}-\frac{xy}{x+y}\right],\quad \Re e[s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}}]>0. \quad (3.1')$$

Example 3.3.2. Consider $f(x)=I_0(2x^{\frac{1}{2}})$. Then

$$\phi(s)=\tfrac{1}{s}\exp(\tfrac{1}{s}),$$
$$G_3(s)=\tfrac{1}{s^2}[1+\tfrac{1}{s}]\exp(\tfrac{1}{s}),$$
$$G_4(s)=\tfrac{2}{s^3}{}_1F_1\left[\begin{matrix}3; \frac{1}{s}\\ 1; s\end{matrix}\right], \quad \Re e\, s>0.$$
$$\gamma(s)=\frac{3\pi^{\frac{1}{2}}}{4(s-1)^{\frac{5}{2}}},$$
$$\sigma(s)=\tfrac{3\pi^{\frac{1}{2}}}{4}s^{-\nu+4}(1-s^2)^{-\frac{7}{2}}[\nu(1-s^2)-5].$$

Therefore using (3.7) yields to the following result

$$\mathcal{L}_n^{-1}\left\{\frac{\nu[p_1{}^2(\overline{s^{\frac{1}{2}}})-1]-5p_1{}^2(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})[p_1{}^2(\overline{s^{\frac{1}{2}}})-1]^{\frac{7}{2}}};\overline{x}\right\}=\frac{4}{3\pi^{\frac{n}{2}}p_n(\overline{x^{\frac{1}{2}}})p_1(\overline{x^{-1}})}$$
$$\cdot\left\{\left(\frac{3\nu-7}{2}+\frac{\nu-2}{p_1(\overline{x^{-1}})}\right)\exp[\frac{1}{p_1(\overline{x^{-1}})}]-4{}_1F_1\left[\begin{matrix}3; & \frac{1}{p_1(\overline{x^{-1}})}\\ 1; & \end{matrix}\right]\right\}, \tag{3.2}$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0$, $n=2,3,\ldots,N$.

Example 3.3.4'. (Two-dimensions)

Assuming n=2 in (3.4), we obtain

$$\mathcal{L}_2^{-1}\left\{\frac{(\nu-5)(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2-\nu}{(s_1s_2)^{\frac{1}{2}}[(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2-1]^{\frac{7}{2}}};x,y\right\}=\frac{2(xy)^{\frac{1}{2}}}{3\pi(x+y)}$$
$$\cdot\left(\frac{(3\nu-7)(x+y)+2(\nu-2)xy}{x+y}\right)\exp\left(\frac{xy}{x+y}\right)-4{}_1F_1\left[\begin{matrix}3; & \frac{xy}{x+y}\\ 1; & \end{matrix}\right]. \tag{3.2'}$$

3.4. The Image of Functions with the Argument $2p_1(\overline{x^{-1}})$

In this section we derive Theorem 3.4.1 involving four different parts. However, we give only the proofs of parts (a) and (b) in detail; the proofs of two other parts are based upon similar ideas, that will not be discussed.

Next we present some direct applications of this theorems. Initially we consider applications on some functions of n variables and obtain their Laplace transformations in n variables. As these functions and their Laplace transforms are usually very complex in nature, we will also consider functions having only two variables and their corresponding Laplace transformations.

Theorem 3.4.1. Let $f(x^2)$ and $x^{-\frac{1}{2}}f(x)$ be of class Ω and let $\phi(s)$ be a one-dimensional Laplace transform of f. Suppose that $x^{-\frac{3}{2}}\phi(\frac{1}{x})$ is also of class Ω. Assume that

(i) $\mathcal{L}\left\{x^{-\frac{j-1}{2}}f(x^2);s\right\}=F_j(s)$ for $j=0,1,2,3,4,5.$

(a) If

(a1) $\mathcal{L}\left\{x^{-\frac{3}{2}}\phi(\frac{1}{x});s\right\}=\xi(s),$

(a2) $\mathcal{L}\left\{x^{-\frac{3}{2}}\xi(\frac{1}{x^2});s\right\}=\alpha(s),$

(a3) $-\frac{d}{ds}\left\{s^{-\nu}\alpha(\frac{1}{s^2})\right\}=\beta(s).$

Let $x^{-\frac{3}{2}}\xi(\frac{1}{x^2})$ be of class Ω and $\frac{d}{ds}\{s^{-\nu}\alpha(\frac{1}{s^2})\}$ exist for $\Re e\, s>c_0$ for some fixed c_0. Moreover, let $x^{-\frac{3}{2}}\exp\left(-sx-\frac{u}{x}\right)f(u)$ and $f(u)x^{-\frac{3}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)u^{-\frac{1}{2}}f(u)$ belong to $L_1\ [(0,\infty)\times(0,\infty)]$.

Then

$$\mathfrak{L}_n\left\{\frac{\nu F_0[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_2[2p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\beta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_1 \text{ for some fixed } \lambda_1, \quad (4.1)$$

provided that the integrals involved exist for $n=2,3,\ldots,N$.

(b) Let us assume the conditions (i) and (a1) and keep $x^{-\frac{3}{2}}\xi(\frac{1}{x^2})$ to be of class Ω, furthermore assume that $x^{-\frac{1}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})u^{-\frac{1}{2}}f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$ and replace (a2) and (a3) by (b1) and (b2) as follows:

(b1) $\mathfrak{L}\left\{x^{\frac{1}{2}}\xi(\frac{1}{x^2});s\right\}=\tau(s)$,

(b2) $-\frac{d}{ds}\left\{s^{-\nu}\tau\left(\frac{1}{s^2}\right)\right\}=\eta(s)$.

Suppose that $\frac{d}{ds}\left\{s^{-\nu}\tau\left(\frac{1}{s^2}\right)\right\}$ exists for $\Re e s>c_1$ for some fixed c_1. Then

$$\mathfrak{L}_n\left\{\frac{(\nu-1)F_1[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_3[2p_1(\overline{x^{-1}})]}{p_n(\overline{x^{-1}})};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n-2}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu}}\eta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_2 \text{ for some fixed } \lambda_2. \quad (4.2)$$

(c) Now assume the condition (i) and replace (a1), (a2) and (a3) by the following conditions:

(c1) $\mathfrak{L}\left\{x^{-\frac{1}{2}}\phi(\frac{1}{x});s\right\}=\gamma(s)$,

(c2) $\mathfrak{L}\left\{x^{-\frac{5}{2}}\gamma(\frac{1}{x^2});s\right\}=\kappa(s)$,

(c3) $-\frac{d}{ds}\left\{s^{-\nu}\kappa(\frac{1}{s^2})\right\}=\theta(s).$

Furthermore, let us assume that $x^{-\frac{5}{2}}\gamma(\frac{1}{x^2})$ is of class Ω and $\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Then

$$\mathfrak{L}_n\left\{\frac{\nu F_2[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_4[2p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\theta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_3 \text{ for some fixed } \lambda_3, \quad (4.3)$$

provided that the integral involved exists for $n=2,3,\dots,N$.

(d) Next suppose the conditions (i) and (c1) remain true, and that $x^{-\frac{5}{2}}\gamma(\frac{1}{x^2})$ is of class Ω and $x^{-\frac{3}{2}}\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Assume $\frac{d}{ds}\left\{s^{-\nu}\delta(\frac{1}{s^2})\right\}$ exists and replace (c2) and (c3) by the following:

(d1) $\mathfrak{L}\left\{x^{-\frac{3}{2}}\gamma(\frac{1}{x^2});s\right\}=\delta(s),$

(d2) $-\frac{d}{ds}\left\{s^{-\nu}\delta(\frac{1}{s^2})\right\}=\zeta(s).$

Then

$$\mathfrak{L}_n\left\{\frac{(\nu-1)F_3[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_5[2p_1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-2}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu}}\zeta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_4 \text{ for some fixed } \lambda_4, \quad (4.4)$$

provided that the integral involved exists and where $n=2,3,\dots,N$. For the existence conditions of n-dimensional Laplace transformations we refer to Brychkov et al. [11; ch. 2].

Proof (a): Form the hypothesis and (a1), we obtain

$$\xi(s)=\int_0^\infty\Big[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{u}{x})f(u)\,du\Big]dx. \tag{4.5}$$

The integrand $x^{-\frac{3}{2}}\exp(-sx-\frac{u}{x})f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, so, by Fubini's Theorem, we interchange the order of integration on the right-side of (4.5).

Hence

$$\xi(s)=\int_0^\infty f(u)\Big[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{u}{x})\,dx\Big]du,\ \Re e\, s>c_0. \tag{4.5'}$$

Next we evaluate the inner integral on the right of (4.5') by using a result from Roberts and Kaufman [87], to get

$$\xi(s)=\pi^{\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}f(u)\exp(-\tfrac{2u^{\frac{1}{2}}}{x})\,du. \tag{4.6}$$

Now we use (4.6) and (a2), to obtain

$$\alpha(s)=\pi^{\frac{1}{2}}\int_0^\infty\Big[\int_0^\infty x^{-\frac{3}{2}}\exp(-sx-\tfrac{2u^{\frac{1}{2}}}{x})u^{-\frac{1}{2}}f(u)\,du\Big]dx. \tag{4.7}$$

Again we evaluate the inner integral on the right side of (4.7) by using a result from Roberts and Kaufman [87], to obtain

$$\alpha(s)=\frac{\pi}{2^{\frac{1}{2}}}\int_0^\infty u^{-\frac{3}{4}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})\,du \tag{4.8}$$

Using (4.8) in (a3), we arrive at

$$s^{\nu+1}\beta(s)=\frac{\pi}{2^{\frac{3}{2}}}\Big\{\nu\int_0^\infty u^{-\frac{3}{4}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{-1})\,du-2^{\frac{3}{2}}\int_0^\infty s^{-1}u^{-\frac{1}{2}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{-1})\,du\Big\},\ \Re e\, s>c_0. \tag{4.9}$$

Now we replace s by $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and multiply both sides of (4.9) by $p_1(\overline{s^{\frac{1}{2}}})$, to obtain

$$\begin{aligned}p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\beta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]&=\frac{\pi}{2^{\frac{1}{2}}}\Big\{\nu\int_0^\infty u^{-\frac{3}{4}}f(u)p_n(\overline{s^{\frac{1}{2}}})\exp[-2^{\frac{3}{2}}u^{\frac{1}{4}}p_1(\overline{s^{\frac{1}{2}}})]\,du\\&-2^{\frac{3}{2}}\int_0^\infty u^{-\frac{1}{4}}f(u)p_n(\overline{s^{\frac{1}{2}}})p_1(\overline{s^{\frac{1}{2}}})\exp[-2^{\frac{3}{2}}u^{\frac{1}{4}}p_1(\overline{s^{\frac{1}{2}}})\,du\Big\}.\end{aligned} \tag{4.10}$$

Next we use the following well-known operational results

$$\begin{aligned}&s_i^{\frac{1}{2}}\exp(-as_i^{\frac{1}{2}})\doteqdot(\pi x_i)^{-\frac{1}{2}}\exp(-\tfrac{a^2}{4x_i}),\\&s_i\exp(-as_i^{\frac{1}{2}})\doteqdot\tfrac{a}{2}\pi^{-\frac{1}{2}}x_i^{-\frac{3}{2}}\exp(-\tfrac{a^2}{4x_i})\ \text{for } i=1,2,\ldots,n.\end{aligned} \tag{*}$$

The relation (4.10) reads as

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\beta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]$$

$$\overset{n}{\underset{n}{=}}\frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}\,p_n(\overline{x^{\frac{1}{2}}})}\Big\{\nu\int_0^\infty u^{-\frac{3}{4}}f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du$$

$$-4p_1(\overline{x^{-1}})\int_0^\infty u^{-\frac{1}{4}}f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du\Big\} \tag{4.11}$$

Let us substitute $u=v^2$ and use (i) for $j=0$ and 2 in (4.11) to obtain

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu-1}\beta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right]$$

$$\overset{n}{\underset{n}{=}}\frac{2^{\frac{1}{2}}}{\pi^{\frac{n-2}{2}}\,p_n(\overline{x^{\frac{1}{2}}})}\cdot\Big\{\nu F_0[2p_1(\overline{x^{-1}})]$$

$$-4p_1(\overline{x^{-1}})F_2[2p_1(\overline{x^{-1}})]\Big\}$$

Thus

$$\mathfrak{L}_n\left\{\frac{\nu F_0[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_2[2p1(\overline{x^{-1}})]}{p_n(\overline{x^{\frac{1}{2}}})};\bar{s}\right\}$$

$$=\frac{\pi^{\frac{n-2}{2}}}{2^{\frac{1}{2}}\,p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu+1}}\beta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_1 \text{ for some constant } \lambda_1.$$

This completes the proof of part (a).

Proof (b): From (4.6) and (b1) it follows that

$$\tau(s)=\pi^{\frac{1}{2}}\int_0^\infty[\int_0^\infty u^{-\frac{1}{2}}f(u)x^{-\frac{1}{2}}\exp(-sx-\tfrac{2u}{x})du]dx. \tag{4.12}$$

The integrand $x^{-\frac{1}{2}}u^{-\frac{1}{2}}f(u)\exp(-sx-\frac{2u^{\frac{1}{2}}}{x})$ belongs to $L_1[(0,\infty)\times(0,\infty)]$, so, by Fubini's Theorem, interchanging the order of the integral on the right of (4.12) is permissible. Next we evaluate the inner integral by using a result from Roberts and Kaufman [87], to obtain

$$\tau(s)=\pi s^{-\frac{1}{2}}\int_0^\infty u^{-\frac{1}{2}}f(u)\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du. \tag{4.13}$$

Plugging (4.13) into (b2), we arrive at

$$s^{\tau}\eta(s)=\pi\left\{\int_0^\infty [(\nu-1)-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{-1}]u^{-\frac{1}{2}}f(u)\exp(-\tfrac{2^{\frac{3}{2}}u^{\frac{1}{4}}}{s})du\right\},\ \Re e\, s>c_1. \tag{4.14}$$

Replacing s by $[p_1(\overline{s^{\frac{1}{2}}})]^{-1}$ and multipling by $p_n(\overline{s^{\frac{1}{2}}})$ both sides of (4.14), and then making use of the operational relations given in (*) from part (a) and using *(i)* for *j=1,3* we arrive at

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{-\nu}\eta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)\right]^n_n \doteq \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}$$
$$\cdot\left\{(\nu-1)F_1[2p_1(x^{-1})]-4p_1(\overline{x^{-1}})F_3[2p_1(\overline{x^{-1}})]\right\}. \tag{4.15}$$

Hence,

$$\mathbf{L}_n\left\{\frac{(\nu-1)F_1[2p_1(\overline{x^{-1}})]-4p_1(\overline{x^{-1}})F_3[2p_1(\overline{x^{-1}})]}{p_n(\overline{x^{-1}})};\bar{s}\right\}$$
$$=\frac{\pi^{\frac{n-2}{2}}}{2p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\nu}}\eta\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^{-1}\right],\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>\lambda_2 \text{ for some fixed } \lambda_2.$$

This completes the proof of part (b).

3.4.1 Applications of Theorem 3.4.1

Example 3.4.1. Consider $f(x)={}_0F_2\left[\begin{matrix} ; \\ \frac{1}{2},1; \end{matrix} -x\right]$. Then

$$\phi(s)=\tfrac{1}{s}\cos(2s^{-\frac{1}{2}}),$$
$$\xi(s)=(\tfrac{\pi}{s})^{\frac{1}{2}}\exp\left(-\frac{1}{s}\right),$$

$$\tau(s)=\frac{\pi}{2^{\frac{5}{2}}}s^{-\nu-3}\exp\left(\frac{s^2}{8}\right)D_{-\frac{3}{2}}\left(\frac{1}{2^{\frac{1}{2}}s^2}\right),$$

$$\eta(s)=\frac{\pi}{2^{\frac{5}{2}}}s^{-\nu-3}\exp\left(\frac{1}{8s^4}\right)\left\{\nu(2^{\frac{1}{2}}s^2)D_{-\frac{3}{2}}\left(\frac{1}{2^{\frac{1}{2}}s^2}\right)-3D_{-\frac{3}{2}}\left(\frac{1}{2^{\frac{1}{2}}s^2}\right)\right\},$$

where $\Re e\, s>0$.

Next, we get

$$F_j(s)=\frac{\Gamma(\frac{j+1}{2})}{s^{\frac{j+1}{2}}}\,{}_2F_2\left[\begin{matrix}\frac{j+1}{4},\frac{j+3}{4}; \\ \frac{1}{2},1\ ;\end{matrix}\ -\frac{4}{s^2}\right],$$

where $\Re e[s+2\exp(\pi ir)]>0$ $(r=0,1)$ and $j=1$ and 3.

Now using (4.2), we arrive at

$$\mathcal{L}_n\left\{\frac{1}{p_n(\overline{x^{\frac{1}{2}}})p_1(\overline{x^{-1}})}\left\{(\nu-1)\,{}_2F_2\left[\begin{matrix}\frac{1}{2},1; \\ \frac{1}{2},1;\end{matrix}\ -\frac{1}{p_1{}^2(\overline{x^{-1}})}\right]-2\,{}_2F_2\left[\begin{matrix}1,\frac{3}{2}; \\ \frac{1}{2},1;\end{matrix}\ -\frac{1}{p_1{}^2(\overline{x^{-1}})}\right]\right\};\bar{s}\right\}$$
$$=\frac{\pi^{\frac{3}{2}}p_1{}^3(\overline{s^{\frac{1}{2}}})}{p_n(\overline{s^{\frac{1}{2}}})}\exp\left[\frac{p_1{}^4(\overline{s^{\frac{1}{2}}})}{8}\right]\left\{\frac{2^{\frac{1}{2}}\nu}{p_1{}^2(\overline{s^{\frac{1}{2}}})}D_{-\frac{3}{2}}\left[\frac{p_1{}^2(\overline{s^{\frac{1}{2}}})}{2^{\frac{1}{2}}}\right]-3D_{-\frac{5}{2}}\left[\frac{p_1{}^2(\overline{s^{\frac{1}{2}}})}{2^{\frac{1}{2}}}\right]\right\},\ \Re e[p_1(\overline{s^{\frac{1}{2}}})]>0. \quad (4.2')$$

Example 3.4.1'. If we let $n=2$, $\nu=3$ in (4.2'), we obtain the following new result in two-dimensions

$$\mathcal{L}_2\left\{\frac{(xy)^{\frac{5}{2}}}{(x+y)^2}\exp\left[-\left(\frac{xy}{x+y}\right)^2\right];s_1,s_2\right\}=$$
$$\frac{\pi(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^3}{(s_1s_2)^{\frac{1}{2}}}\exp\left[\frac{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^4}{\cdot 8}\right]\left\{\frac{2^{\frac{1}{2}}}{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2}D_{-\frac{3}{2}}\left[\frac{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2}{2^{\frac{1}{2}}}\right]-D_{-\frac{5}{2}}\left[\frac{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2}{2^{\frac{1}{2}}}\right]\right\}. \quad (4.2'')$$

Example 3.4.2. Assume $f(x)={}_0F_3\left[\begin{matrix} ; \\ 1,\frac{1}{2},;\end{matrix}\ -x\right]$. Then

$$\phi(s)=\frac{1}{s}\,{}_1F_3\left[\begin{matrix} ; \\ 1,1; \end{matrix}-\frac{1}{s}\right],$$

$$\xi(s)=\left(\frac{\pi}{s}\right)^{\frac{1}{2}}{}_2F_3\left[\begin{matrix} 1\ ; \\ 1,1,\frac{3}{2}; \end{matrix}-\frac{1}{s}\right],$$

$$\tau(s)=\frac{\pi}{2s^{\frac{3}{2}}}\,{}_4F_3\left[\begin{matrix} 1,\frac{1}{2},\frac{3}{4},\frac{5}{4}; \\ 1,1,\frac{3}{2}\ ; \end{matrix}-\left(\frac{2}{s}\right)^2\right] \text{where, } \Re e\, s>0.$$

For $v=1$, we obtain

$$\eta(s)=-\frac{\pi s}{4}\left\{4\,{}_4F_3\left[\begin{matrix} 1,\frac{1}{2},\frac{3}{4},\frac{5}{4}; \\ 1,1,\frac{3}{2}\ ; \end{matrix}-4s^2\right]+5s^2\,{}_4F_3\left[\begin{matrix} 2,\frac{3}{2},\frac{7}{4},\frac{9}{4}; \\ 2,2,\frac{5}{2}\ ; \end{matrix}-4s^2\right]\right\},\ \Re e\, s>0.$$

Also,

$$F_3(s)=\frac{1}{s}J_0\left(\frac{4}{s}\right),\ \Re e\, s>0.$$

Now for $v=1$ from (4.2), we arrive at

$$\mathcal{L}_n\left\{\frac{1}{\overline{p_1(x^{-1})}}J_0\left[\frac{2}{\overline{p_1(x^{-1})}}\right];\bar{s}\right\}$$

$$=\frac{\pi}{8p_n(\overline{s^{\frac{1}{2}}})p_1{}^2(\overline{s^{\frac{1}{2}}})}\left\{4\,{}_4F_3\left[\begin{matrix} 1,\frac{1}{2},\frac{3}{4},\frac{5}{4}; \\ 1,1,\frac{3}{2}\ ; \end{matrix}-\frac{4}{p_1{}^2(\overline{s^{\frac{1}{2}}})}\right]+5\,{}_4F_3\left[\begin{matrix} 2,\frac{1}{2},\frac{3}{2},\frac{5}{2}; \\ 2,2,\frac{5}{2}\ ; \end{matrix}-\frac{4}{p_1{}^2(\overline{s^{\frac{1}{2}}})}\right]\right\}, \quad (4.2.1)$$

where $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0,\ n=2,3,\ldots,N.$

Remark 3.4.1: If we choose n=2, then we deduce the following new result in two-dimensions.

$$\mathcal{L}_2\left\{\frac{xy}{x+y}J_0\left(\frac{2xy}{x+y}\right);s_1,s_2\right\}=\frac{1}{8[(s_1s_2)(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})]^2}$$

$$\cdot\left\{4\,{}_4F_3\left[\begin{matrix} 1,\frac{1}{2},\frac{3}{4},\frac{5}{4}; \\ 1,1,\frac{3}{2}\ ; \end{matrix}\frac{4}{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2}\right]+5\,{}_4F_3\left[\begin{matrix} 2,\frac{3}{2},\frac{7}{4},\frac{9}{4}; \\ 2,2,\frac{5}{2}\ ; \end{matrix}\frac{4}{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^2}\right]\right\}. \quad (4.2.2)$$

3.5. The Original of Functions with the Argument $[p_1(\overline{s^{\frac{1}{2}}})]^2$

This section begins with Theorem 3.5.1 that involves two parts. The proof of part (a) that is more complicated than that of part (b) is given in detail, so that part (b) is stated without proof. However, this part can be proved by an analogous process to that employed for developing the result in part (a).

Next, some new inversions formula have been obtained by applying the results of this Theorem to the most commonly used special functions. Initially we consider applications on some special functions of n variables. As these functions and their inverse Laplace transformations are usually very complex in nature, we will provide also two-dimensional inverse Laplace transformations having only two variables. Moreover, we will use some of these two-dimensional inversion formulas in Chapter 4 for solving a certain type of non-homogenous linear partial differential equations.

Theorem 3.5.1. Suppose that $f(x^2)$ is of class Ω and let $\phi(s)$ be a one-dimensional Laplace transform of f. Assume that $x^{\frac{1}{2}}\phi(\frac{1}{x})$ is also of class Ω. Let

(i) $\mathbf{L}\left\{x^j f(x^2);s\right\} = H_j(s)$ for $j = 1,2,3.$

Moreover, let $x^{-\frac{7}{2}}\xi(\frac{1}{x^2})$ be of class Ω and $x^{\frac{1}{2}}\exp[-sx-\frac{u}{x}]f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$.

(a) Assume that

(a1) $\mathbf{L}\left\{x^{\frac{1}{2}}\phi(\frac{1}{x});s\right\} = \xi(s),$

(a2) $\mathbf{L}\left\{x^{-\frac{5}{2}}\xi(\frac{1}{x^2});s\right\} = \gamma(s).$

Let $x^{-\frac{1}{2}}\exp[-sx-\frac{2u^{\frac{1}{2}}}{x}]u^{-\frac{1}{2}}f(u)$ belong to $L_1[(0,\infty)\times(0,\infty)]$. Then

$$\mathbf{L}^{-1}{}_{n}\left\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^2\right];\overline{x}\right\} = \frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}$$
$$\cdot\left\{\tfrac{1}{2}H_1[2p_1(\overline{x^{-1}})] + p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})] + 4p_1^{\,2}(\overline{x^{-1}})H_3[2p_1(\overline{x^{-1}})]\right\}, \qquad (5.1)$$

provided that the integrals involved on the left side of (5.1) exist for $n = 2,3,\ldots,N$.

(b) Assume the conditions (i) and (a1), and replace (a2) by

(b1) $\mathcal{L}\left\{x^{-\frac{7}{2}}\xi(\frac{1}{x^2});s\right\} = \eta(s)$.

Keep on the hypotheses that $x^{-\frac{7}{2}}\xi(\frac{1}{x^2})$ is of class Ω and $x^{-\frac{1}{2}}\exp[-sx - \frac{2u^{\frac{1}{2}}}{x}]u^{\frac{1}{2}}f(u)$ belongs to $L_1[(0,\infty)\times(0,\infty)]$. Then

$$\mathcal{L}_n^{-1}\left\{\frac{\overline{p_1(s^{\frac{1}{2}})}}{p_n(s^{\frac{1}{2}})}\eta\left[\left(\overline{p_1(s^{\frac{1}{2}})}\right)^2\right];\bar{x}\right\} = \frac{1}{\pi^{\frac{n-2}{2}}\,\overline{p_n(x^{\frac{1}{2}})}}$$
$$\cdot\left\{H_1[2\overline{p_1(x^{-1})}] + 2\overline{p_1(x^{-1})}H_2[2\overline{p_1(x^{-1})}]\right\}. \tag{5.2}$$

It is assumed that the integrals involved exist for $n = 2,3,\ldots,N$.

Proof (a): We begin with using the definition of one-dimensional Laplace transform for f and making use of (a1), to obtain

$$\begin{aligned}\phi(s) &= \int_0^\infty \exp(-su)f(u)du,\\ \xi(s) &= \int_0^\infty\left[\int_0^\infty x^{\frac{1}{2}}\exp(-sx - \tfrac{u}{x})f(u)du\right]dx, \quad \Re e\, s > c_0 \text{ for some fixed } c_0.\end{aligned} \tag{5.3}$$

Next we wish to interchange the order of integrations, (an operation which is valid,by Fubini's Theorem because $x^{\frac{1}{2}}\exp[-sx - \frac{u}{x}]f(u) \in L_1[(0,\infty)\times(0,\infty)]$.

Thus

$$\xi(s) = \int_0^\infty\left[\int_0^\infty x^{\frac{1}{2}}\exp(-sx - \tfrac{u}{x})dx\right]f(u)du, \quad \Re e\, s > c_0. \tag{5.4}$$

Making use of an operational relation from Roberts and Kaufman [87] in (5.4) yields

$$\xi(s) = \frac{\pi^{\frac{1}{2}}}{2}s^{-\frac{3}{2}}\int_0^\infty f(u)[1 + 2u^{\frac{1}{2}}s^{\frac{1}{2}}]\exp(-2u^{\frac{1}{2}}s^{\frac{1}{2}})du. \tag{5.5}$$

If we substitute (5.5) into (a2) a little algebra leads to

$$\gamma(s)=\frac{\pi^{\frac{1}{2}}}{2}\left\{\int_0^\infty\left[\int_0^\infty x^{\frac{1}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)f(u)du\right]dx\right.$$
$$\left.+2\int_0^\infty\left[\int_0^\infty x^{\frac{1}{2}}\exp\left(-sx-\frac{2u^{\frac{1}{2}}}{x}\right)u^{\frac{1}{2}}f(u)du\right]dx\right\}. \tag{5.6}$$

In a similar manner, which is discussed to change (5.3) to (5.5) yields

$$s^{\frac{3}{2}}\gamma(s)=\frac{\pi}{4}\left\{\int_0^\infty[1+2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}}]\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})f(u)du\right.$$
$$\left.+4\int_0^\infty s\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})u^{\frac{1}{2}}f(u)du\right\}. \tag{5.7}$$

Now, we replace s by $[p_1(\overline{s^{\frac{1}{2}}})]^2$ and multiply both sides of (5.7) by $p_n(\overline{s^{\frac{1}{2}}})$. It follows that

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^{\frac{3}{2}}\gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^2\right]=\frac{\pi}{4}\left[\int_0^\infty f(u)p_n(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du+2^{\frac{3}{2}}\int_0^\infty f(u)u^{\frac{1}{4}}p_n(\overline{s^{\frac{1}{2}}})\right.$$
$$\left.p_1(\overline{s^{\frac{1}{2}}})\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du\right]+\pi\int_0^\infty f(u)u^{\frac{1}{2}}p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^2\exp(-2^{\frac{3}{2}}u^{\frac{1}{4}}s^{\frac{1}{2}})du. \tag{5.8}$$

Next, a tedious calculation as we did in the proof of Theorem 3.3.1, leads to

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^3\gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^2\right]$$
$$\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{s^{\frac{1}{2}}})}\left\{\int_0^\infty f(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du\right.$$
$$+2p_1{}^2(\overline{x^{-1}})\int_0^\infty uf(u)\exp[-2u^{\frac{1}{2}}p_1(\overline{x^{-1}})]du \tag{5.9}$$

Upon substituting $u=v^2$ into (5.9) and then using (i) for $j=1,2,3$, we arrive at

$$p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^3\gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^2\right]\overset{n}{\underset{n}{=}}\frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}$$
$$\cdot\left\{\frac{1}{2}H_1[2p_1(\overline{x^{-1}})]+p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})]+4p_1{}^2(\overline{x^{-1}})H_3[2p_1(\overline{x^{-1}})]\right\}.$$

Therefore

$$\mathfrak{L}_n^{-1}\left\{\frac{[p_1(\overline{s^{\frac{1}{2}}})]^3}{p_n(\overline{s^{\frac{1}{2}}})}\gamma\left[\left(p_1(\overline{s^{\frac{1}{2}}})\right)^2\right];\overline{x}\right\}=\frac{1}{\pi^{\frac{n-2}{2}}p_n(\overline{x^{\frac{1}{2}}})}$$

$$\cdot\left\{\tfrac{1}{2}H_1[2p_1(\overline{x^{-1}})]+p_1(\overline{x^{-1}})H_2[2p_1(\overline{x^{-1}})]+4p_1^2(\overline{x^{-1}})H_3[2p_1(\overline{x^{-1}})]\right\}, \quad (5.1')$$

where $n=2,3,\ldots,N$

3.5.1. Example Based Upon Theorem 3.5.1

Example 3.5.1. Assume that $f(x)={}_qF_p\left[\begin{matrix}(a)_p;\\(b)_q;\end{matrix}cx\right]$. Then

$$\phi(s)=\tfrac{1}{s}\,{}_qF_p\left[\begin{matrix}(a)_p,1;\frac{c}{s}\\(b)_q\;;\end{matrix}\right],\ \text{where } q\geq p,\ \Re e\, s>|\Re e\, c|,$$

$$\xi(s)=\frac{3\pi^{\frac{1}{2}}}{4s^{\frac{5}{2}}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2};\frac{c}{s}\\(b)_q\quad;\end{matrix}\right],\ \text{where } \Re e\, s>0 \text{ if } p+1<q, \Re e\, s>\Re e\, c \text{ if } p+1=q,$$

$$\gamma(s)=\frac{45\pi}{32s^{\frac{7}{2}}}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2},\frac{7}{4},\frac{9}{4};\\(b)_q\quad;\end{matrix}\right],$$

where $p+4\leq q+1$, $\Re e\, s>0$ if $p+4\leq q$, $\Re e[s+2c\exp(\pi ir)]>0 (r=0,1)$ if $p+4=q+1$

Now, we calculate

$$H_j(s)=\frac{\Gamma(j+1)}{s^{j+1}}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{j+1}{2},\frac{j+2}{2};\frac{4c}{s^2}\\(b)_q\quad;\end{matrix}\right]\ \text{for } j=1,2,3\ ,$$

and where $p+2\leq q+1$, $\Re e\, s>0$ if $p+2\leq q$, $\Re e[s+2j\exp(\pi ir)]>0 (r=0,1)$ if $p+2=q+1$

Therefore, using (5.1) we arrive at

$$\mathfrak{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^4}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2},\frac{7}{4},\frac{9}{4};\ \frac{c}{p_1^4(\overline{s^{\frac{1}{2}}})}\\(b)_q\quad;\end{matrix}\right];\overline{x}\right\}$$

$$\mathcal{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^4}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2},\frac{7}{4},\frac{9}{4}; & \dfrac{c}{p_1{}^4(\overline{s^{\frac{1}{2}}})}\\ (b)_q & ;\end{matrix}\right];\bar{x}\right\}$$

$$=\frac{8}{9\pi^{\frac{3}{2}}p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^2}\left\{\tfrac{1}{2}\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,1,\frac{3}{2}; & \dfrac{c}{p_1^2(\overline{x^{-1}})}\\ (b)_q & ;\end{matrix}\right]+{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{3}{2},2; & \dfrac{c}{p_1(\overline{x^{-1}})}\\ (b)_q & ;\end{matrix}\right]\right.$$

$$\left.+6\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,2,\frac{5}{2}; & \dfrac{c}{p_1^2(\overline{x^{-1}})}\\ (b)_q & ;\end{matrix}\right]\right\}, \tag{5.1a}$$

where $p+4\le q+1$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})]>0$ if $p+4\le q$, $\Re e[p_1(\overline{s^{\frac{1}{2}}})+2c\exp(\pi ir)]>o$ $(r=0,1)$ if $p+3=q$ and $n=2,3,\ldots,N$.

Similarly, if we use (5.2) we obtain the following result

$$\mathcal{L}_n^{-1}\left\{\frac{1}{p_n(\overline{s^{\frac{1}{2}}})[p_1(\overline{s^{\frac{1}{2}}})]^4}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2},\frac{5}{4},\frac{7}{4}; & \dfrac{c}{p_1{}^4(\overline{s^{\frac{1}{2}}})}\\ (b)_q & ;\end{matrix}\right];\bar{x}\right\}$$

$$=\frac{4}{9\pi^{\frac{3}{2}}p_n(\overline{x^{\frac{1}{2}}})[p_1(\overline{x^{-1}})]^2}\left\{{}_{p+2}F_q\left[\begin{matrix}(a)_p,1,\frac{3}{2}; & \dfrac{c}{p_1^2(\overline{x^{-1}})}\\ (b)_q & ;\end{matrix}\right]+2\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{3}{2},2; & \dfrac{c}{p_1^2(\overline{x^{-1}})}\\ (b)_q & ;\end{matrix}\right]\right\}. \tag{5.1b}$$

Remark 3.5.1: Choosing n=2 and c=1 in (5.3.b), it follows that

$$\mathcal{L}_2^{-1}\left\{\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^4}\,{}_{p+4}F_q\left[\begin{matrix}(a)_p,1,\frac{5}{2},\frac{5}{4},\frac{7}{4}; & \dfrac{4}{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})^4}\\ (b)_q & ;\end{matrix}\right]\right\}$$

$$=\frac{4(xy)^{\frac{3}{2}}}{9\pi(x+y)^2}\left\{{}_{p+2}F_q\left[\begin{matrix}(a)_p,1,\frac{3}{2}; & \left(\dfrac{xy}{x+y}\right)^2\\ (b)_q & ;\end{matrix}\right]+2\,{}_{p+2}F_q\left[\begin{matrix}(a)_p,\frac{3}{2},2; & \left(\dfrac{xy}{x+y}\right)^2\\ (b)_q & ;\end{matrix}\right]\right\}. \tag{5.1b'}$$

CHAPTER 4. THE SOLUTION OF INITIAL-BOUNDARY-VALUE PROBLEMS (IBVP'S) BY DOUBLE LAPLACE TRANSFORMATIONS

4.1. Introduction

One of the basic properties of the Laplace transform is that it transforms the operation of taking a derivative into the operation of mere multiplication by a variable. It thus replaces some of the derivatives in a partial differential equation (PDE) (depending on which variables are involved in the transform) by multiples of the function, and therefore (hopefully) reduces the equation to an easier equation to solve. For example, a PDE in two independent variables will sometimes, after application of the Laplace transform, become an ordinary differential equation (ODE). The Laplace transform is not capable of simplifying most PDEs, but it does help in some cases.

Basically, the idea behind any transform (not just the Laplace transformation) is this. We have a difficult problem to solve. We apply a transform to it and produce an easier problem. We solve this easier problem. Then we have to transform the solution of the easier problem back into the solution of the original harder problem. This last step is one where most of the difficulty arises. Inverting the transform may be quite difficult. For the Laplace transformation it usually requires the use of contour integration in the complex plane. However there are many problems whose solution may be found in terms of Laplace transforms for which the inversion using the techniques of complex analysis is too complicated.

By the use of multiple Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in N independent complex variables. This algebraic equation can be solved for the multiple transform of the solution of the original PDE. The multiple inversion of this transform then gives the desired solution. The analytic difficulty of evaluating multiple inverse transforms increases with the number of independent variables. This still requires a significant amount of effective techniques of contour integration for deriving the final solution. With the help of tables given in Voelker and Doetsch [107] Ditkin and Prudnikov [43] Hlaidk [52] Dahiya [21], [23], [27], [28], [29] and [30] and our results in Chapters 2 and 3, the actual evaluation of the inversion integral is alleviated.

Several IBVPs characterized by non-homogenous PDEs are explicitly solved in this chapter by means of some of our results developed in Chapters 2 and 3. We shall confine ourselves to the case that the transform is taken with respect to two variables. In the absence of necessitous three and N-dimensional Laplace transformation tables, several IBVPs characterized by PDEs are explicitly solved by double Laplace transformations. These include non-homogenous linear PDEs of the first order, non-homogenous second order PDEs of Hyperbolic type and non-homogenous second order linear PDEs of Parabolic type.

We would like to remark that calculations made in this chapter are formal. However, we are given the conditions for which the transform equations and the inverse transform equations exist. At the end one can verify that all the Laplace transform equations performed are valid. Indeed, it can be checked the original function $u(x,y)$ is such that all previously used operations are permissible. We denote $\mathbf{L}_2\{u(x,y);s_1,s_2\}=U(s_1,s_2)$ and $\mathbf{L}_2^{-1}\{U(s_1,s_2);x,y\}=u(x,y)$ through this chapter.

4.2. Non-homogenous Linear Partial Differential Equations (PDEs) of the First Order

4.2.1. Partial Differential Equations of Type

$$u_x+u_y=f(x,y),\ 0<x<\infty,\ 0<y<\infty \tag{4.2.1.1}$$

under boundary conditions

$$\begin{aligned} u(x,0)&=\alpha(x),\\ u(0,y)&=\beta(y). \end{aligned} \tag{4.2.1.2}$$

Example 4.2.1.1. Determination of a solution $u=u(x,y)$ of (4.2.1.1) and (4.2.1.2) for

(a) $f(x,y)=(x+y)^{\frac{1}{2}},\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=\exp(-x)$ and $\beta(y)=\exp(-y)$.

(b) $f(x,y)=(x+y)^{-\frac{1}{2}},\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=x^{\frac{1}{2}}$ and $\beta(y)=y^{\frac{1}{2}}$.

(c) $f(x,y)=\dfrac{xy}{(x+y)^{\frac{3}{2}}},\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=0=\beta(y)$.

(d) $f(x,y)=(x+y)^{\frac{3}{2}}-(x^{\frac{3}{2}}+y^{\frac{3}{2}}),\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=\exp(-x)$ and $\beta(y)=\exp(-y)$.

(e) $f(x,y)=\dfrac{(xy)^n}{(x+y)^{n+\frac{3}{2}}},\ n\in\mathrm{N}$ and $0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=0=\beta(y)$.

We first present the solutions of parts (a) and (e) in detail, and next the outline of the solutions of other parts will be provided.

(a) Taking the two-dimensional Laplace transform from (4.2.1.1a) and (4.2.1.2a) with the aid of (1.5.2.7), (1.5.2.9) and Remark 2.1.1.2 Equation (1.1'''), yields

$$s_1U(s_1,s_2)-\frac{1}{s_2+1}+s_2U(s_1,s_2)-\frac{1}{s_1+1}=\frac{\pi^{\frac{1}{2}}(s_1+s_2+s_1^{\frac{1}{2}}s_2^{\frac{1}{2}})}{2(s_1s_2)^{\frac{3}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}$$

and so,

$$U(s_1,s_2)=\frac{s_1+s_2+2}{(s_1+1)(s_2+1)(s_1+s_2)}+\frac{\pi^{\frac{1}{2}}(s_1+s_2+s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{2(s_1s_2)^{\frac{3}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \tag{1.1}$$

Now, the inversion of (1.1) can be obtained by using (1.5.2.28)

$$u(x,y)=\exp[-(x+y)]+2\begin{cases}\int_0^x \exp[-(x+y)+2\xi]d\xi & \text{if } y>x\\ \int_0^y \exp[-(x+y)+2\xi]d\xi & \text{if } y<x\end{cases}$$
$$+\pi^{\frac{1}{2}}\begin{cases}\int_0^x (x+y-2\xi)^{\frac{1}{2}}d\xi & \text{if } y>x\\ \int_0^y (x+y-2\xi)^{\frac{1}{2}}d\xi & \text{if } y<x\end{cases}. \tag{1.2}$$

After evaluating the integrals and doing the elementary calculation it is found that

$$u(x,y)=\begin{cases}\exp[-(y-x)]+\frac{\pi^{\frac{1}{2}}}{6}[(y-x)^{\frac{3}{2}}-(x+y)^{\frac{3}{2}}] & \text{if } y>x\\ \exp[-(x-y)]+\frac{\pi^{\frac{1}{2}}}{6}[(x-y)^{\frac{3}{2}}-(x+y)^{\frac{3}{2}}] & \text{if } y<x\end{cases}.$$

(e) Applying the two-dimensional Laplace transformation to (4.2.1.1e) and (4.2.1.2e) with the help of (1.5.2.7), (1.5.2.9), and formula 181 in Brychkov et al. [11; p.300], we obtain

$$(s_1+s_2)U(s_1,s_2)=\frac{\pi\Gamma(2n+1)}{2^{2n}\Gamma(n+\frac{3}{2})}\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2n+1}}.$$

Hence,

$$U(s_1,s_2)=\frac{\pi\Gamma(2n+1)}{2^{2n}\Gamma(n+\frac{3}{2})}\cdot\frac{1}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2n+1}(s_1+s_2)}.$$

Making use of (1.5.2.28), we obtain that

$$u(x,y)=\begin{cases}\displaystyle\int_0^x\frac{(x-\xi)^n(y-\xi)^n}{(x+y-2\xi)^{n+\frac{3}{2}}}d\xi & \text{if } y>x\\ \displaystyle\int_0^y\frac{(x-\xi)^n(y-\xi)^n}{(x+y-2\xi)^{n+\frac{3}{2}}}d\xi & \text{if } y<x\end{cases}.$$

Evaluating the integrals after tedious calculation the solution $u(x,y)$ turns out to be

$$u(x,y)=\begin{cases}\displaystyle\frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{(n-2k-\frac{1}{2})}\left[(x-y)^{2k}(x+y)^{n-2k-\frac{1}{2}}-\ (y-x)^{n-\frac{1}{2}}\right] & \text{if } y>x\\ \displaystyle\frac{1}{2^{2n+1}}\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{(n-2k-\frac{1}{2})}\left[(y-x)^{2k}(x+y)^{n-2k-\frac{1}{2}}-\ (x-y)^{n-\frac{1}{2}}\right] & \text{if } y<x\end{cases}$$

Now, with the aid of correspondence results given in Chapters 2 and 3 the similar procedure we have followed, we get the following transform equations, for parts (b), (c) and (d)

(b) $U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}}{2}\left[\frac{s_1^{\frac{3}{2}}+s_2^{\frac{3}{2}}}{(s_1s_2)^{\frac{3}{2}}(s_1+s_2)}\right]+\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)},$

(c) $U(s_1,s_2)=\frac{1}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^3(s_1+s_2)},$

(d) $U(s_1,s_2)=\frac{s_1+s_2+2}{(s_1+1)(s_2+1)(s_1+s_2)}+\frac{3\pi^{\frac{1}{2}}}{4(s_1s_2)^{\frac{3}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)},$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Therefore,

(b) $u(x,y) = (x+y)^{\frac{1}{2}}$.

$$\textbf{(c)}\ u(x,y) = \begin{cases} \frac{2}{3}(y-x)^{\frac{3}{2}} - \frac{1}{6}(x+y)^{\frac{3}{2}} - \frac{1}{2}\dfrac{(y-x)^2}{(x+y)^{\frac{1}{2}}} & \text{if } y > x \\ \frac{2}{3}(x-y)^{\frac{3}{2}} - \frac{1}{6}(x+y)^{\frac{3}{2}} - \frac{1}{2}\dfrac{(x-y)^2}{(x+y)^{\frac{1}{2}}} & \text{if } y < x \end{cases}.$$

$$\textbf{(d)}\ u(x,y) = \tfrac{1}{5}\begin{cases} 5\exp[-(y-x)] + [(y-x)^{\frac{5}{2}} + (x+y)^{\frac{5}{2}} - 2(x^{\frac{5}{2}} + y^{\frac{5}{2}})] & \text{if } y > x \\ 5\exp[-(x-y)] + [(x-y)^{\frac{5}{2}} + (x+y)^{\frac{5}{2}} - 2(x^{\frac{5}{2}} + y^{\frac{5}{2}})] & \text{if } y < x \end{cases}.$$

Example 4.2.1.2. Determination of a solution $u = u(x,y)$ of (4.2.1.1) and (4.2.1.2) for

(a) $f(x,y) = \dfrac{(xy)^{\tau-1}(y-x)}{(x+y)^{\tau+\frac{1}{2}}}$, $0 < x < \infty$, $0 < y < \infty$, $\Re e\ \tau > 0$ and

$\alpha(x) = 0 = \beta(y)$.

(b) $f(x,y) = \dfrac{(xy)^{\tau-1}(y-x)}{(x+y)^{\tau+\frac{3}{2}}}$, $0 < x < \infty$, $0 < y < \infty$, $\Re e\ \tau > 0$ and

$\alpha(x) = 0 = \beta(y)$.

We discuss only the solution of part (a) in detail; the solution of part (b) is provided in brief because these parts are based upon similar techniques.

(a) Taking the two-dimensional Laplace transformation of each term of (4.2.1.1a) and (4.2.1.2a) by the use of (1.5.2.7), (1.5.2.9), Example 1.1.1 equation (1.1') and formula (1.5.2.20), we get

$$(s_1 + s_2)U(s_1, s_2) = \pi^{\frac{1}{2}}\Gamma(\tau+1)\left[\frac{1}{s_2^{\frac{1}{2}}}\int_{s_1}^{\infty}\frac{1}{\lambda^{\frac{1}{2}}(\lambda^{\frac{1}{2}} + s_2^{\frac{1}{2}})^{2\tau+1}}d\lambda - \frac{1}{s_1^{\frac{1}{2}}}\int_{s_2}^{\infty}\frac{1}{\lambda^{\frac{1}{2}}(s_1^{\frac{1}{2}} + \lambda^{\frac{1}{2}})^{2\tau+1}}d\lambda\right]$$

Evaluating the integral, yields

$$(s_1+s_2)U(s_1,s_2)=\frac{\pi\Gamma(\tau+1)}{\tau}\cdot\frac{s_1-s_2}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\tau+1}},\ \Re e\ \tau>0.$$

Hence,

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}\Gamma(\tau+1)}{\tau}\left[\frac{s_1}{(s_1+s_2)(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\tau+1}}-\frac{s_2}{(s_1+s_2)(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\tau+1}}\right],$$

where $\Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$, $\Re e\ \tau>0$.

Now, making use of (1.5.2.31), we obtain the following inverse transform

$$u(x,y)=\begin{cases}\dfrac{\tau(xy)^{\tau}}{(x+y)^{\tau+\frac{1}{2}}}-\tau\displaystyle\int_0^x[(x-\xi)(y-\xi)]^{\tau-1}\left\{\tau[(x-\xi)^2+(y-\xi)^2]-(x-\xi)(y-\xi)\right\} \\ \qquad\cdot(x+y-2\xi)^{-\tau-\frac{3}{2}}d\xi \text{ if } y<x \\ -\dfrac{\tau(xy)^{\tau}}{(x+y)^{\tau+\frac{1}{2}}}-\tau\displaystyle\int_0^y[(x-\xi)(y-\xi)]^{\tau-1}\left\{\tau[(x-\xi)^2+(y-\xi)^2]-(x-\xi)(y-\xi)\right\} \\ \qquad .(x+y-2\xi)^{-\tau-\frac{3}{2}}d\xi \text{ if } y>x\end{cases},\ \Re e\ \tau>0. \tag{2.1}$$

Remark 4.2.1.1: If we let $\tau=1$ in (4.2.1.1a), we obtain

$$u_x+u_y=\frac{y-x}{(x+y)^{\frac{3}{2}}},$$

$$\alpha(x)=0=\beta(y).$$

Hence, from (2.1) we get

$$u(x,y)=\begin{cases}\dfrac{xy}{(x+y)^{\frac{3}{2}}}-\displaystyle\int_0^x[(x-y)^2+(x-\xi)(y-\xi)](x+y-2\xi)^{-\frac{5}{2}}d\xi & \text{if } y>x \\ -\dfrac{xy}{(x+y)^{\frac{3}{2}}}+\displaystyle\int_0^y[(x-y)^2+(x-\xi)(y-\xi)](x+y-2\xi)^{-\frac{5}{2}}d\xi & \text{if } y<x\end{cases}$$

Evaluating the integrals, we arrive at

$$u(x,y)=\begin{cases}\dfrac{xy}{(x+y)^{\frac{3}{2}}}+\dfrac{x-y}{4}\left\{\frac{4}{3}(y-x)^{\frac{1}{2}}-\frac{1}{3}(y-x)^{2}(x+y)^{-\frac{3}{2}}-(x+y)^{\frac{1}{2}}\right\} & \text{if } y>x\\ -\dfrac{xy}{(x+y)^{\frac{3}{2}}}-\dfrac{x-y}{4}\left\{\frac{4}{3}(x-y)^{\frac{1}{2}}-(x-y)^{2}(x+y)^{-\frac{3}{2}}-(x+y)^{\frac{1}{2}}\right\} & \text{if } y<x.\end{cases}$$

(b) By an analogous argument, we arrive at the following transform equation

$$U(s_1,s_2)=\frac{\pi\Gamma(2\tau+1)}{\tau 2^{2\tau}\Gamma(\tau+\frac{3}{2})}\cdot\frac{s_1-s_2}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\tau+1}},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0,\ \Re e\,\tau>0. \tag{2.2}$$

The inverse transform of (2.2) can be obtained using Formula (1.5.2.31) and the well known result in Brychkov et al. [11]

$$u(x,y)=\begin{cases}\dfrac{\tau(xy)^{\tau}}{(x+y)^{\tau+\frac{3}{2}}}-\tau\displaystyle\int_0^x[(x-\xi)(y-\xi)]^{\tau-1}\{\tau[(x-\xi)^2+(y-\xi)^2] \\ \qquad -3(x-\xi)(y-\xi)](x+y-2\xi)^{-\tau-\frac{5}{2}}\}d\xi \quad \text{if } y>x \\ -\dfrac{\tau(xy)^{\tau}}{(x+y)^{\tau+\frac{3}{2}}}+\tau\displaystyle\int_0^y[(x-\xi)(y-\xi)]^{\tau-1}\{\tau[(x-\xi)^2+(y-\xi)^2] \\ \qquad -3(x-\xi)(y-\xi)](x+y-2\xi)^{-\upsilon-\frac{5}{2}}\}d\xi \quad \text{if } y<x,\end{cases},$$

where $\Re e\,\tau>0$.

Example 4.2.1.3. Determination of a solution $u=u(x,y)$ of (4.2.1.1) and (4.2.1.2) for

$$f(x,y)=\frac{1}{(x+y)^{\frac{1}{2}}}\exp(-\frac{xy}{x+y}),\ 0<x<\infty,\ 0<y<\infty \text{ and}$$

$$\alpha(x)=\sinh x \text{ and } \beta(y)=\sinh y.$$

We begin to solve part (b) completely, and next we give the outline of the solution of part (a).

We apply the two-dimensional Laplace transformation to (4.2.1.1b) and (4.2.1.2.b) with the aid of (1.5.2.7) and (1.5.2.9) and Remark 3.2.3 Equation (2.2''') with (1.5.2.20), to get

$$(s_1+s_2)U(s_1,s_2)=\frac{1}{(s_1^{\,2}-1)}+\frac{1}{(s_2^{\,2}-1)}+\frac{\pi^{\frac{1}{2}}}{s_2^{\frac{1}{2}}}\int_{s_1}^{\infty}\frac{(\lambda^{\frac{1}{2}}+s_2^{\frac{1}{2}})d\xi}{\lambda^{\frac{1}{2}}[1+(\lambda^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^2}-\frac{\pi^{\frac{1}{2}}}{s_1^{\frac{1}{2}}}\int_{s_2}^{\infty}\frac{(s_1^{\frac{1}{2}}+\lambda^{\frac{1}{2}})d\xi}{\lambda^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+\lambda^{\frac{1}{2}})^2]^2}.$$

Evaluating the integrals and simplfying, we arrive at

$$U(s_1,s_2)=\frac{s_1+s_2}{(s_1^{\,2}-1)(s_2^{\,2}-1)}-\frac{2(s_1s_2+1)}{(s_1^{\,2}-1)(s_2^{\,2}-1)(s_1+s_2)}$$
$$-\frac{\pi^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1+s_2)(s_1s_2)^{\frac{1}{2}}[(1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]},\ \Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \qquad (3.1)$$

The inverse of (3.1) can be obtained using Remark 3.2.3 Equation (2.2''') with (1.5.2.20) and Example 3.2.3' Equation (3.5'') and Formula (1.49) and (1.59) in Ditkin and Prudnikov [43; pages, 106 and 107]

$$u(x,y)=\sinh(x-y)-2\begin{cases}\int_0^x\cosh(x+y-2\xi)d\xi-\frac{1}{2}\int_0^x\frac{1}{(x+y-2\xi)^{\frac{1}{2}}}\exp[-\frac{(x-\xi)(y-\xi)}{x+y-2\xi}]d\xi \text{ if } y>x\\ \int_0^y\cosh(x+y-2\xi)d\xi-\frac{1}{2}\int_0^y\frac{1}{(x+y-2\xi)^{\frac{1}{2}}}\exp[-\frac{(x-\xi)(y-\xi)}{x+y-2\xi}d\xi \text{ if } y<x.\end{cases}$$

Example 4.2.1.4. Determination of a solution $u=u(x,y)$ of (4.2.1.1) and (4.2.1.2) for

$$f(x,y)=\frac{1}{(x+y)^{\frac{1}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2};\\ 1;\end{matrix}-\frac{xy}{x+y}\right],$$
$$\alpha(x)=\sin x \text{ and } \beta(y)=\sin y. \qquad (4.1)$$

Taking the two-dimensional Laplace transformation of each term of (4.1), using (1.5.2.7), (1.5.2.9) and Remark 3.2.2 Equation (2.3'), we obtain

$$(s_1+s_2)U(s_1,s_2)-\frac{1}{s_2^{\,2}+1}-\frac{1}{s_1+1}=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1+s_2)^2]^{\frac{3}{2}}}, \qquad \Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$$

Hence,

$$U(s_1,s_2)=\frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}+2}{(s_1^{\frac{1}{2}}+1)+(s_2^{\frac{1}{2}}+1)(s_1+s_2)}+\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}(s_1+s_2)}.$$

Now, using formula (1.4.8) and (1.5.8) in Ditkin and Prudnikov [43; p.106] and Formula (2.3') in Remark 3.2.2 with the aid of (1.5.2.28), we arrive at the following solution

$$u(x,y)=\sin(x+y)-2\begin{cases}\int_0^x\cos(x+y-2x)dx+\int_0^x\frac{1}{(x+y-2x)^{\frac{1}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix}-\frac{(x-x)(y-x)}{(x+y-2x)}\right]dx & if\, y>x\\ \int_0^x\cos(x+y-2x)dx+\int_0^y\frac{1}{(x+y-2x)^{\frac{1}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix}-\frac{(x-x)(y-x)}{(x+y-2x)}\right]dx & if\, y<x.\end{cases}$$

Substituting $x+y-2\xi=t$, we obtain

$$u(x,y)=\begin{cases}\sin(y-x)+\frac{1}{2}\int_{x+y}^{y-x}t^{-\frac{1}{2}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix}-\frac{t^2-(x-y)^2}{t}\right]dt & \text{if}\, y>x\\ \sin(x-y)+\frac{1}{2}\int_{x+y}^{x-y}t^{-\frac{1}{2}}\,{}_1F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix}-\frac{t^2-(x-y)^2}{t}\right]dt & \text{if}\, y<x.\end{cases}$$

4.2.2. Partial Differential Equations of Type

$$au_x+bu_y+\varepsilon cu=f(x,y),\ 0<x<\infty,\ 0<y<\infty \qquad (4.2.2.1)$$

where a, b and c are constants, such that $c>0$ and $\frac{b}{a}>0$ and $\varepsilon=\pm 1$.

Subject to boundary conditions

$$u(x,0)=\alpha(x),\ u(0,y)=\beta(y) \tag{4.2.2.2}$$

Example 4.2.2.1. Determination of a solution $u=u(x,y)$ of (4.2.2.1) and (4.2.2.2) for

(a) $f(x,y)=(x+y)^{-\frac{1}{2}}$,and

$\alpha(x)=0=\beta(y)$,

where $\varepsilon=+1$.

(b) $f(x,y)=(x+y)^{-\frac{3}{2}}$ and

$\alpha(x)=0=\beta(y)$,

where $\varepsilon=+1$

We discuss the solution of part (a) in detail; the solution of part (b) is provided in brief because the two parts are based upon similar techniques.

(a) Applying the two-dimensional Laplace transformation to (4.2.2.1a) and (4.2.2.2a) with the help of (1.5.2.7), (1.5.2.9) and Remark 2.1.1.2 Equation (1.1"), we obtain

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(as_1+bs_2+c)},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0 \tag{1.1}$$

With the aid of formula (1.5.2.30) and Remark 2.1.1.2 Equation (1.1"), we get

$$u(x,y)=\begin{cases}\frac{1}{a}\int_0^x \frac{\exp(-\frac{c}{a}\xi)}{[(x+y)-(1+\frac{b}{a})\xi]^{\frac{1}{2}}}d\xi & \text{if } y>x\\ \frac{1}{b}\int_0^y \frac{\exp(-\frac{c}{b}\eta)}{[(x+y)-(1+\frac{b}{a})\eta]^{\frac{1}{2}}}d\eta & \text{if } y<x\end{cases}$$

Evaluating the integrals, we arrive at the following solution

$$u(x,y)=\begin{cases} a[\frac{\pi}{c(a+b)}]^{\frac{1}{2}}\exp[\frac{-c(x+y)}{a+b}]\left\{Erfi\left([\frac{c(x+y)}{a+b}]^{\frac{1}{2}}\right)-Erfi\left([\frac{c(ay-bx)}{a(a+b)}]^{\frac{1}{2}}\right)\right\} & \text{if } y>x \\ b[\frac{\pi}{c(a+b)}]^{\frac{1}{2}}\exp[\frac{-c(x+y)}{a+b}]\left\{Erfi\left([\frac{c(x+y)}{a+b}]^{\frac{1}{2}}\right)-Erfi\left([\frac{c(bx-ay)}{b(a+b)}]^{\frac{1}{2}}\right)\right\} & \text{if } y<x, \end{cases}$$

where $a>0,\ b>0$ and $c>0$.

(b) Similarly, with the help of Formula 181 in Brychkov et al. [11], we arrive at the following transform equation

$$U(s_1,s_2)=\frac{2\pi^{\frac{1}{2}}}{(as_1+bs_2+c)(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})},\quad \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0 \tag{1.2}$$

Using Formula (1.5.2.30), we obtain

$$u(x,y)=\begin{cases} \frac{2a}{a+b}\exp\left[\frac{c(x+y)}{a+b}\right]\left\{\left(\frac{a}{ay-bx}\right)^{\frac{1}{2}}\exp\left[\frac{-c(ay-bx)}{a(a+b)}\right]\right. \\ -\frac{1}{(x+y)^{\frac{1}{2}}}\exp\left[\frac{-c(x+y)}{a+b}\right]+\left(\frac{\pi c}{a+b}\right)^{\frac{1}{2}}\left\{Erf\left[\left(\frac{c(ay-bx)}{a(a+b)}\right)^{\frac{1}{2}}\right]\right. \\ \left.\left.-Erf\left[\left(\frac{c(x+y)}{a+b}\right)^{\frac{1}{2}}\right]\right\}\right\} & \text{if } y>x \\ \frac{2b}{a+b}\exp\left[\frac{c(x+y)}{a+b}\right]\left\{\left(\frac{b}{bx-ay}\right)^{\frac{1}{2}}\exp\left[\frac{-c(bx-ay)}{b(a+b)}\right]\right. \\ -\frac{1}{(x+y)^{\frac{1}{2}}}\exp\left[\frac{-c(x+y)}{a+b}\right]+\left(\frac{\pi c}{a+b}\right)^{\frac{1}{2}}\left\{Erf\left[\left(\frac{c(bx-ay)}{b(a+b)}\right)^{\frac{1}{2}}\right]\right. \\ \left.\left.-Erf\left[\left(\frac{c(x+y)}{a+b}\right)^{\frac{1}{2}}\right]\right\}\right\} & \text{if } y<x\ , \end{cases}$$

where $a>0,\ b>0,$ and $c>0$.

Remark 4.2.2.1:

(a) If we let $a=b=c=1$ in part (a), in fact for the following boundary value problem

$$u_x+u_y+u=\frac{1}{(x+y)^{\frac{1}{2}}},\quad 0<x<\infty,\ 0<y<\infty$$

$\alpha(x) = 0 = \beta(y)$,

we obtain a solution as follows

$$u(x,y) = (\tfrac{\pi}{2})^{\frac{1}{2}} \exp[-\frac{x+y}{2}] \begin{cases} Erfi\left[\left(\frac{x+y}{2}\right)^{\frac{1}{2}}\right] - Erfi\left[\left(\frac{y-x}{2}\right)^{\frac{1}{2}}\right] & \text{if } y > x \\ Erfi\left[\left(\frac{x+y}{2}\right)^{\frac{1}{2}}\right] - Erfi\left[\left(\frac{x-y}{2}\right)^{\frac{1}{2}}\right] & \text{if } y < x \end{cases}.$$

(b) Substituting $a = b = c = 1$ and $\varepsilon = -1$ in the equation (4.2.2.1), yields

$u_x + u_y - u = \dfrac{1}{(x+y)^{\frac{3}{2}}}$, $0 < x < \infty$, $0 < y < \infty$ and

$\alpha(x) = 0 = \beta(y)$.

so that,

$$u(x,y) = \exp(\frac{x+y}{2}) + \begin{cases} \frac{1}{(y-x)^{\frac{1}{2}}}\exp\left[-\frac{(y-x)}{2}\right] - \frac{1}{(y-x)^{\frac{1}{2}}}\exp\left[-\frac{x+y}{2}\right] \\ +(\frac{\pi}{2})^{\frac{1}{2}}\left\{Erf\left[\left(\frac{y-x}{2}\right)^{\frac{1}{2}}\right] - Erf\left[\left(\frac{x+y}{2}\right)^{\frac{1}{2}}\right]\right\} & \text{if } y > x \\ \frac{1}{(x-y)^{\frac{1}{2}}}\exp\left[-\frac{(x-y)}{2}\right] - \frac{1}{(x-y)^{\frac{1}{2}}}\exp\left[-\frac{x+y}{2}\right] \\ +(\frac{\pi}{2})^{\frac{1}{2}}\left\{Erf\left[\left(\frac{x-y}{2}\right)^{\frac{1}{2}}\right] - Erf\left[\left(\frac{x+y}{2}\right)^{\frac{1}{2}}\right]\right\} & \text{if } y < x \end{cases}$$

4.3. Non-homogenous Second order Linear Partial Differential Equations of Hyperbolic Type

4.3.1. Partial Differential Equations of Type

$$u_{xy} = f(x,y),\ 0 < x < \infty,\ 0 < y < \infty \text{ and} \qquad (4.3.1.1)$$

$$u(x,0) = \alpha(x),\ u(0,y) = \beta(y),\ u(0,0) = c, \text{ where } c \text{ is a constant} \qquad (4.3.1.2)$$

Example 4.3.1.1. Determination of a solution $u = u(x,y)$ of (4.3.1.1) and (4.3.1.2) for

(a) $f(x,y) = (x+y)^{\frac{1}{2}} - y^{\frac{1}{2}},\ 0 < x < \infty,\ 0 < y < \infty$ and

$\alpha(x) = x^{\mu},\ \beta(y) = y^{\nu}$ and $c = 0$.

(b) $f(x,y) = (x+y)^{-\frac{3}{2}},\ 0 < x < \infty,\ 0 < y < \infty$ and

$\alpha(x) = x^{\mu},\ \beta(y) = y^{\nu}$ and $c = 0$.

First we begin to solve the part (a) of the problem. Due to similarity of the procedure, the details of the solution of the BVP in part (b) are omitted.

(a) We apply the two-dimensional Laplace transform to (4.3.1.1a) and (4.3.1.2a) using (1.5.2.11) and Formula (1.1') in Example 1.1.1 for $\nu = 0$ with the aid of Operational relation (35) in Voelker and Doetsch [107; p. 185] we obtain the following transform equation

$$U(s_1,s_2) = \frac{\Gamma(\mu+1)}{{s_1}^{\mu+1}s_2} + \frac{\Gamma(\nu+1)}{s_1{s_2}^{\frac{1}{2}}} + \frac{\pi^{\frac{1}{2}}}{({s_1}^{\frac{1}{2}} + {s_2}^{\frac{1}{2}})},\ \Re e\ [{s_1}^{\frac{1}{2}} + {s_2}^{\frac{1}{2}}] > 0.$$

Now, using (1.5.2. 32), yields

$$u(x,y) = x^{\mu} + y^{\nu} + 4[x^{\frac{1}{2}} + y^{\frac{1}{2}} - (x+y)^{\frac{1}{2}}],\ \Re e\ (\mu,\nu) > -1$$

(b) With the similar process as employed in part (a), we arrive at the following transform equation

$$U(s_1,s_2) = \frac{\Gamma(\mu+1)}{{s_1}^{\mu+1}s_2} + \frac{\Gamma(\nu+1)}{s_1{s_2}^{\nu+1}} + \frac{\pi^{\frac{1}{2}}}{2{s_1}^{\frac{5}{2}}{s_2}^{\frac{3}{2}}({s_1}^{\frac{1}{2}} + {s_2}^{\frac{1}{2}})},\ \Re e[{s_1}^{\frac{1}{2}} + {s_2}^{\frac{1}{2}}] > 0.$$

Next, we apply Relation (1.5.2.32), to obtain

$$u(x,y)=x^{\mu}+y^{\nu}+\tfrac{4}{15}[(x+y)^{\frac{5}{2}}-y^{\frac{5}{2}}-\tfrac{5}{2}xy^{\frac{3}{2}}-x^{\frac{5}{2}}],\ \Re e\ (\mu,\nu)>-1.$$

Example 4.3.1.2. Determination of a solution $u=u(x,y)$ (4.3.1.1) and (4.3.1.2) for

(a) $f(x,y)=\dfrac{(xy)^{\frac{\nu}{2}}}{(x+y)^{\frac{\nu+1}{2}}},\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=x^{\mu},\ \beta(y)=y^{\nu}$, and $c=0$.

(b) $f(x,y)=\dfrac{(xy)^{\frac{\nu}{2}}}{(x+y)^{\frac{\nu+3}{2}}},\ 0<x<\infty,\ 0<y<\infty$ and

$\alpha(x)=x^{\mu},\ \beta(y)=y^{\nu}$, and $c=0$.

We solve the boundary value problem given in part (a) completely, the solution of part (b) is similarly direct.

(a) Making use of the double Laplace transformation on each term of (4.3.1.1a) and (4.3.1.2a), yields the following transform equation

$$s_1s_2U(s_1,s_2)=\frac{\Gamma(\mu+1)s_1}{s_1^{\mu+1}}+\frac{\Gamma(\tau+1)s_2}{s_2^{\tau+1}}+\frac{\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2}+1)}{(s_1s_2)^{\frac{1}{2}}(s_1+s_2)^{\nu+1}}$$

by virtue of relation (1.5.2.11) and the result of Example 1.1.1 Equation (1.1').

Hence,

$$U(s_1,s_2)=\frac{\Gamma(\mu+1)}{s_1^{\mu+1}s_2}+\frac{\Gamma(\tau+1)}{s_1s_2^{\tau+1}}+\frac{\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2}+1)}{(s_1s_2)^{21}(s_1+s_2)^{\nu+1}},\ \Re e\ [s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0,\ \Re e\ (\mu,\tau,\nu)>-1.$$

Further, if we apply the Relation (1.5.2.32) to the transform equation, we find that

$$u(x,y)=x^{\mu}+y^{\tau}+\int_0^x\int_0^y\frac{(\xi\eta)}{(\xi+\eta)^{\frac{\nu+1}{2}}}d\xi d\eta, \tag{1.1}$$

where, $\Re e\,\mu > -1$, $\Re e\,\tau > -1$, and $\Re e\,\nu > -1$

Remark 4.3.1.1: If we let ν be a fixed even number, say $2n$, then we can reduce the solution (1.1) to the form

$$u(x,y) = x^{\mu} + y^{\tau} + \int_0^x \int_0^y \frac{(\xi\eta)^n}{(\xi+\eta)^{n+\frac{1}{2}}} d\xi d\eta,$$

where, $\Re e\,\mu > -1$, $\Re re\,\tau > -1$

Evaluating the integrals, after tedious calculation the solution (1.1) in this case may be written in the following form

$$u(x,y) = x^{\mu} + y^{\tau} + 4\sum_{k=0}^{n}\left\{\tfrac{(-1)^{k+1}}{2k-1}\binom{n}{k}\left\{-\tfrac{1}{3}x^{\frac{3}{2}} + \sum_{p=0}^{k}\tfrac{(-1)^{p+1}}{2k-3}\binom{k}{p}\left[(x+y)^{\frac{3}{2}-p}y^{p} - y^{\frac{3}{2}}\right]\right\}\right\}, \tag{2.1}$$

where, $\Re e\,\mu > 0$, $\Re e\,\tau > 0$.

(b) The solution of part (b) can be derived in a similar fashion, using the Formula 181 in Brychkov et al. [11], yields the transform equation

$$U(s_1,s_2) = \frac{\Gamma(\mu+1)}{s_1^{\mu+1}s_1} + \frac{\Gamma(\tau+1)}{s_1 s_2^{\tau+1}} + \frac{\pi\Gamma(\nu+1)}{2^{\nu}\Gamma(\frac{\nu+3}{2})}\cdot\frac{1}{(ss)(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\nu+1}},\ \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}] > 0,\ \Re e\,(\mu,\tau,\nu) > -1$$

Hence,

$$u(x,y) = x^{\mu} + y^{\tau} + \int_0^x \int_0^y \frac{(\xi\eta)^{\frac{\nu}{2}}}{(\xi+\eta)^{\frac{\nu+3}{2}}} d\xi d\eta, \tag{1.3}$$

where, $\Re e\,\mu > -1$, $\Re e\,\tau > -1$

Indeed, putting $\nu = 2(n-1)$, a tedious calculation, yields

$$u(x,y) = x^{\mu} + y^{\tau} + 4\sum_{k=0}^{n-1}\left\{\frac{(-1)^{k-1}}{2k+1}\binom{n-1}{k}\left[\frac{y^{n-\frac{1}{2}}}{1-2n} + (x+y)^{n-k+\frac{1}{2}}.\sum_{p=0}^{n+k-1}(-1)^{p+1}\binom{n+k-1}{p}\left(y^{p} - y^{n+p-k-\frac{1}{2}}\right)\right]\right\},$$

where, $\Re e\,\mu > -1$, $\Re e\,\tau > -1$ (1.4)

Example 4.3.1.1'. If we let $\nu = 2n = 0$ in parts (a) and (b) of Example 4.3.1.1, respectively the problem reduces to

(a') $u_{xy} = \dfrac{1}{(x+y)^{\frac{1}{2}}}$

$$\alpha(x) = x^{\mu},\ \beta(y) = y^{\tau},\ c = 0.$$

(b') $u_{xy} = \dfrac{1}{(x+y)^{\frac{3}{2}}}$

$$\alpha(x) = x^{\mu},\ \beta(y) = y^{\tau},\ c = 0.$$

Then, we obtain the following solutions for each part, respectively

$$u(x,y) = x^{\mu} + y^{\tau} + \tfrac{4}{3}[(x+y)^{\frac{3}{2}} - (x^{\frac{3}{2}} + y^{\frac{3}{2}})].$$

$$u(x,y) = x^{\mu} + y^{\tau} + 4[x^{\frac{1}{2}} + y^{\frac{1}{2}} + -(x+y)^{\frac{1}{2}}].$$

Furthermore, for $\nu = 2(n-1) = 2$, the problem in part (b) reduces to the following

$$u_{xy} = \frac{xy}{(x+y)^{\frac{5}{2}}}$$

$$\alpha(x) = x^{\mu}, \beta(y) = y^{\tau}, c > 0.$$

which has a solution

$$u(x,y) = x^{\mu} + y^{\tau} + \frac{2xy}{3(x+y)^{\frac{3}{2}}} \quad x > 0, y > 0.$$

Example 4.3.1.2. Determination of a solution $u = u(x,y)$ of (4.3.1.1) and (4.3.1.2) for

$$f(x,y) = \frac{xy}{(x+y)^{\frac{5}{2}}}\,{}_1F_1\left[\begin{matrix}\frac{5}{2}; & -\dfrac{xy}{x+y} \\ 2; & \end{matrix}\right], 0 < x < \infty, 0 < y < \infty \text{ and}$$

$$\alpha(x) = x^{\mu}, \beta(y) = y^{\tau}, c = 0, \qquad \Re e(\mu, \tau) > 0. \tag{1.1}$$

We apply the two-dimensional Laplace transformation on each term of (1.1), to get the following transform equation

$$s_1 s_2 U(s_1, s_2) = \frac{\Gamma(\mu+1)s_1}{s_1^{\mu+1}} + \frac{\Gamma(\tau+1)s_2}{s_2^{\tau+1}} + \frac{\pi}{2[1+(s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}})^2]^{\frac{3}{2}}},$$

by virtue of Relation (1.5.2.11) and the Result (2.2.i) of Example 2.2.2'.
Therefore

$$U(s_1, s_2) = \frac{\Gamma(\mu+1)}{s_1^{\mu+1} s_2} + \frac{\Gamma(\tau+1)}{s_1 s_2^{\tau+1}} + \frac{\pi}{2 s_1 s_2 [1+(s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}})]^{\frac{3}{2}}},$$

$\Re e[s_1^{\frac{1}{2}} + s_2^{\frac{1}{2}}] > 0$, $\Re e(\mu, \tau) > -1$.

Furthermore, if we apply the Relation (1.5.2.32) to the transform equation, we find that

$$u(x,y) = x^{\mu} + y^{\tau} + \frac{(xy)^{\frac{1}{2}}}{x+y} \exp\left(\frac{xy}{x+y}\right), \quad x > 0, \ y > 0 \text{ and } \Re e(\mu, \tau) > -1$$

4.3.2. The Wave Equation

To illustrate the use of some of our results which are derived in Chapters 2 and 3 in *wave mechanics*, we shall consider the one-dimensional wave equation in a normalized form

$$u_{xx} - u_{yy} = f(x,y), \quad 0 < x < \infty, 0 < y < \infty. \tag{4.3.2.1}$$

The initial and boundary conditions are

$$\begin{aligned} &u(x,0) = \alpha(x), \ u(0,y) = \beta(y) \\ &u_y(x,0) = \theta(x), \ u_x(0,y) = \delta(y), \text{and} \\ &\alpha(0) = \beta(0), \end{aligned} \tag{4.3.2.2}$$

subject to the following condition of compatibility:

$$\int_0^x f(x-\tau,\tau)d\tau+\delta(x)-\theta(x)+\frac{\partial}{\partial x}[\beta(x)-\alpha(x)]=0. \qquad (4.3.2.3)$$

Example 4.3.2.1. Determination of a solution $u=u(x,y)$ of (4.3.2.1) and (4.3.2.2) for

(a) $f(x,y)=0$

$\alpha(x)=x^{\frac{1}{2}}$, $\beta(y)=y^{\frac{1}{2}}$

$\theta(x)=\frac{1}{2}x^{-\frac{1}{2}}$ and $\delta(y)=\frac{1}{2}y^{-\frac{1}{2}}$.

(b) $f(x,y)=y^n-x^n$

$\alpha(x)=x^{\frac{1}{2}}, \beta(y)=y^{\frac{1}{2}}$

$\theta(x)=\frac{1}{2}x^{-\frac{1}{2}}$ and $\delta(y)=\frac{1}{2}y^{-\frac{1}{2}}$.

(c) $f(x,y)=x^{\nu-2}y^{\nu}-x^{\nu}y^{\nu-2}, \Re e\, \nu>1$

$\alpha(x)=x^{\frac{1}{2}}, \beta(y)=y^{\frac{1}{2}}$

$\theta(x)=\frac{1}{2}x^{-\frac{1}{2}}$ and $\delta(y)=\frac{1}{2}y^{-\frac{1}{2}}$.

We provide the solution for part (c). By an argument similar to that employed in part (c) the solution for the two other parts are established in brief.

(c) Taking the two-dimensional Laplace transformation from each term of (4.3.2.1a) and (4.3.2.2a) with the aid of (1.5.2.8) and (1.5.2.10), yields the following transform equation

$$(s_1^{\,2}-s_2^{\,2})U(s_1,s_2)-\frac{\Gamma(\frac{3}{2})s_1}{s_2^{\frac{3}{2}}}-\frac{\Gamma(\frac{3}{2})}{s_2^{\frac{1}{2}}}+\frac{\Gamma(\frac{3}{2})s_2}{s_1^{\frac{3}{2}}}+\frac{\Gamma(\frac{3}{2})}{s_1^{\frac{1}{2}}}$$
$$=\frac{\Gamma(\nu-1)\Gamma(\nu+1)}{s_1^{\,\nu-1}s_2^{\,\nu+1}}-\frac{\Gamma(\nu-1)\Gamma(\nu+1)}{s_1^{\,\nu+1}s_2^{\,\nu-1}}$$

or, equivalently

$$U(s_1,s_2)=\Gamma(\tfrac{3}{2})\left[\frac{s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}}{(s_1s_2)^{\frac{3}{2}}}-\frac{1}{s_1s_2(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}\right]+\frac{\Gamma(\nu-1)\Gamma(\nu+1)}{s_1^{\nu+1}s_2^{\nu+1}},\quad \Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \quad (1.1)$$

For the inversion of (1.1) we combine Remark 2.1.1.2 Equation (1.1''') with Formula 87 in Voelker and Doetsch [107: p. 236]. Finally, we obtain the solution in the form

$$u(x,y)=(x+y)^{\frac{1}{2}}+\frac{1}{\nu(\nu-1)}x^{\nu}y^{\nu},\quad \Re e\ \nu>1. \qquad (1.2)$$

Proceeding in the same way as in the establishment of solution (1.2), we can then show that the transform equations for parts (a) and (b) are as follow, respectively

$$U(s_1,s_2)=\frac{\Gamma(\frac{3}{2})(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{3}{2}}}-\frac{1}{s_1s_2(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}$$

$$U(s_1,s_2)=\frac{\Gamma(\frac{3}{2})(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{(s_1s_2)^{\frac{3}{2}}}-\frac{1}{s_1s_2(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}+\frac{\sum\limits_{\substack{i,j=1\\ i<j}}^{n-1}s_is_j}{(s_1s_2)^{n+1}(s_1+s_2)}$$

Therefore,

$$u(x,y)=(x+y)^{\frac{1}{2}}.$$

$$u(x,y)=(x+y)^{\frac{1}{2}}+\begin{cases}\dfrac{xy^{n+1}}{\Gamma(1)\Gamma(n)}{}_2F_1\left[\begin{matrix}1,-1; & \frac{y}{x}\\ n+2; & x\end{matrix}\right]+\dfrac{x^2y^n}{\Gamma(2)\Gamma(n-1)}{}_2F_1\left[\begin{matrix}1,-2; & \frac{y}{x}\\ n+1; & x\end{matrix}\right] & \\ +\dots+\dfrac{x^{n+1}y}{\Gamma(n)\Gamma(1)}{}_2F_1\left[\begin{matrix}1,-n\ ; & \frac{y}{x}\\ 3\ ; & x\end{matrix}\right] & \text{if } y>x\\ \dfrac{x^{n+1}y}{\Gamma(n)\Gamma(1)}{}_2F_1\left[\begin{matrix}1,-n; & \frac{x}{y}\\ 3\ ; & y\end{matrix}\right]+\dfrac{x^ny^2}{\Gamma(n-1)\Gamma(2)}{}_2F_1\left[\begin{matrix}1,1-n; & \frac{x}{y}\\ 4\ ; & y\end{matrix}\right] & \\ +\dots+\dfrac{xy^{n+1}}{\Gamma(n)\Gamma(1)}{}_2F_1\left[\begin{matrix}1,-1; & \frac{x}{y}\\ n+2; & y\end{matrix}\right] & \text{if } y<x,\end{cases}$$

by virtue of the Results in Remark 2.1.1.2 Equation (1.1''') and the following inversion formula given in the unpublished monograph by R. S. Dahiya

$$L_2^{-1}\{s_1^{k-\mu-1}s_2^{k-\nu-1}(s_1+s_2)^{-k};x,y\}$$
$$=\begin{cases} \dfrac{x^{\mu-k}y^{\nu}}{\Gamma(\mu-k)\Gamma(\nu+1)}{}_2F_1\left[\begin{matrix} k,k-\mu; \dfrac{y}{x} \\ \nu+1; \end{matrix}\right] \text{ if } y>x \\ \dfrac{x^{\mu}y^{\nu-k}}{\Gamma(\mu+1)\Gamma(\nu-k)}{}_2F_1\left[\begin{matrix} k,k-\nu; \dfrac{x}{y} \\ \mu+1; \end{matrix}\right] \text{ if } y<x, \end{cases} \quad \mu,\nu>-1.$$

Example 4.3.2.2. Determination of a solution $u=u(x,y)$ of (4.3.2.1) and (4.3.2.2) for

(a) $f(x,y)=\dfrac{x^{\nu-1}y^{\nu}-x^{\nu}y^{\nu-1}}{(x+y)^{\nu+\frac{1}{2}}}$ and
$\alpha(x)=0=\beta(y)$,
$\theta(x)=x^{-\frac{1}{2}}$ and $\delta(y)={}^{-\frac{1}{2}}$.

(b) $f(x,y)=\dfrac{x^{\nu-1}y^{\nu}-x^{\nu}y^{\nu-1}}{(x+y)^{\nu+\frac{3}{2}}}$ and
$\alpha(x)=0=\beta(y)$,
$\theta(x)=x^{-\frac{1}{2}}$ and $\delta(y)=y^{-\frac{1}{2}}$.

We only present the solution of part (b) completely, the solution of part (a) is similar to that of part (b), hence the details of that have been omitted.

(b) Applying the double Laplace transform to the terms of (4.3.2.1b) and (4.3.2.2b) with the aid of (1.5.2.8), (1.5.2.10), (1.5.2.20) and Formula 181 in Brychkov et al. [11; p. 300], finally we arrive at

$$(s_1-s_2)U(s_1,s_2)-\frac{\pi^{\frac{1}{2}}}{s_2^{\frac{1}{2}}}+\frac{\pi^{\frac{1}{2}}}{s_1^{\frac{1}{2}}}=\frac{\pi\Gamma(2\nu+1)}{2^{2\nu}\Gamma(\nu+\frac{3}{2})}\left\{\int_{s_1}^{\infty}\frac{d\lambda}{(\lambda^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{2\nu+1}}-\int_{s_2}^{\infty}\frac{d\lambda}{(s_1^{\frac{1}{2}}+\lambda^{\frac{1}{2}})^{2\nu+1}}\right\}$$

Evaluating the integrals and simplifying, we get

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)}+\frac{\pi\Gamma(2\nu+1)}{\nu 2^{2\nu}\Gamma(\nu+\frac{3}{2})}\cdot\frac{s_1^{\frac{1}{2}}-s_2^{\frac{1}{2}}}{(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)}, \qquad (2.1)$$

where $\Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

The inversion of (2.1) will be accomplished from (1.5.28), and Relations (1.1') in Example 1.1.1 and Formula 181 in Brychkov et al.

$$u(x,y)=\frac{1}{\nu}\begin{cases}\displaystyle\int_0^x\frac{\nu(x+y-2\xi)^{\nu+1}+[(x-\xi)(y-\xi)]^\nu}{(x+y-2\xi)^{\nu+\frac{3}{2}}}d\xi & \text{if } y>x\\ \displaystyle\int_0^y\frac{\nu(x+y-2\xi)^{\nu+1}+[(x-\xi)(y-\xi)]^\nu}{(x+y-2\xi)^{\nu+\frac{3}{2}}}d\xi & \text{if } y<x.\end{cases} \qquad (2.2)$$

Substituting $x+y-2\xi=t$ in (2.2), we obtain

$$u(x,y)=\begin{cases}(x+y)^{\frac{1}{2}}-(y-x)^{\frac{1}{2}}-\frac{1}{\nu 2^{2\nu+1}}\displaystyle\int_{x+y}^{y-x}[t^2-(x-y)^2]^\nu t^{-\nu-\frac{3}{2}}dt & \text{if } y>x\\ (x+y)^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}-\frac{1}{\nu 2^{2\nu+1}}\displaystyle\int_{x+y}^{x-y}[t^2-(x-y)^2]^\nu t^{-\nu-\frac{3}{2}}dt & \text{if } y<x, \quad \Re e\,\nu>0.\end{cases} \qquad (3.2)$$

(a) Similarly, the transform equation for part (a) is as follows

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}}{(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)}+\frac{\pi^{\frac{1}{2}}\Gamma(\nu+1)}{\nu(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})(s_1+s_2)}, \quad \Re e\,[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0.$$

The inversion of $U(s_1,s_2)$ will be accomplished from (1.5.28) and Relation (1.1') in Example 1.1.1

$$u(x,y)=\frac{1}{\nu}\begin{cases}\displaystyle\int_0^x\frac{\nu(x+y-2\xi)^{\nu}+[(x-\xi)(y-\xi)]^\nu}{(x+y-2\xi)^{\nu+\frac{1}{2}}}d\xi & \text{if } y>x\\ \displaystyle\int_0^y\frac{\nu(x+y-2\xi)^{\nu}+[(x-\xi)(y-\xi)]^\nu}{(x+y-2\xi)^{\nu+\frac{1}{2}}}d\xi & \text{if } y<x,\end{cases}$$

Remark 4.3.2.1: One may easily check that the condition of compatibility (4.3.2.3) holds true for all IBVPs given in Example 4.3.2.1.

Remark 4.3.2.2: If we let $\nu = 1$, in part (a) and part (b), we obtain the following solutions, respectively

$$u(x,y)=\begin{cases}(x+y)^{\frac{1}{2}}-(y-x)^{\frac{1}{2}}-\frac{1}{12}[(y-x)^{\frac{3}{2}}-(x+y)^{\frac{3}{2}}] \\ \qquad -\frac{1}{4}(x-y)^2[(y-x)^{-\frac{1}{2}}-(x+y)^{-\frac{1}{2}}] & \text{if } y>x \\ (x+y)^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}-\frac{1}{12}[(x-y)^{\frac{3}{2}}-(x+y)^{\frac{3}{2}}] \\ \qquad -\frac{1}{4}(x-y)^2[(x-y)^{-\frac{1}{2}}-(x+y)^{-\frac{1}{2}}] & \text{if } y<x.\end{cases}$$

$$u(x,y)=\begin{cases}\frac{5}{4}[(x+y)^{\frac{1}{2}}-(y-x)^{\frac{1}{2}}]-\frac{1}{12}(x-y)^2[(y-x)^{-\frac{3}{2}}-(x+y)^{-\frac{3}{2}}] & \text{if } y>x \\ \frac{5}{4}[(x+y)^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}]-\frac{1}{12}(x-y)^2[(x-y)^{-\frac{3}{2}}-(x+y)^{-\frac{3}{2}}] & \text{if } y<x.\end{cases}$$

Example 4.3.2.3. Determination of a solution $u = u(x,y)$ of (4.3.2.1) and (4.3.2.2) for

(a) $f(x,y)=\dfrac{(xy)(x-y)}{(x+y)^{\frac{3}{2}}}\exp\left(-\dfrac{xy}{x+y}\right)$, and
$\alpha(x)=\exp(-x),\quad \beta(y)=\exp(-y)$
$\theta(x)=0=\delta(y).$

(b) $f(x,y)=\dfrac{x-y}{(x+y)^{\frac{5}{2}}}\,{}_2F_1\left[\begin{matrix}\frac{5}{2}; \\ 2;\end{matrix}-\dfrac{xy}{x+y}\right]$, and
$\alpha(x)=0=\beta(y)$
$\theta(x)=\exp(-x),\quad \delta(y)=\exp(-y).$

Let us begin to solve the IBVP in part (a). Taking the two-dimensional Laplace transformation of each term of (4.3.2.1a) and (4.3.2.2.a), with the help of (1.5.2.8) and (1.5.2.10), and also using the Relation (3.5') in Example 3.2.3' and with the aid of (1.5.2.20), we arrive at

$$(s_1-s_2)U(s_1,s_2)=\frac{(s_1{}^2-s_2{}^2)+(s_1-s_2)}{(s_1+1)(s_2+1)}+\frac{\pi(s_1{}^{\frac{1}{2}}-s_2{}^{\frac{1}{2}})}{(s_1s_2)^{\frac{1}{2}}}\int_{(s_1{}^{\frac{1}{2}}+s_2{}^{\frac{1}{2}})}^{\infty}\frac{t}{(1+t^2)^2}dt$$

or, equivalently

$$U(s_1,s_2)=\frac{1}{(s_1+1)(s_2+1)}+\frac{1}{(s_1+1)(s_2+1)(s_1+s_2)}+\frac{\pi}{2(s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})](s_1+s_2)}$$

Making use of Relations (1.2) in Ditkin and Prudnikov[43; p.100, (201)] together with Relation (201) in Brychkov et al. [11; p. 303]. Finally with the aid of (1.5.2.28), we obtain the following solution

$$u(x,y)=\exp[-(x-y)]+\begin{cases}\int_0^x \exp[-(x+y-2\xi)]d\xi+\int_0^x \frac{1}{(x+y-2\xi)^{\frac{1}{2}}}\gamma(1,\frac{(x-\xi)(y-\xi)}{(x+y-2\xi)})d\xi & \text{if } y>x\\ \int_0^y \exp[-(x+y-2\xi)]d\xi+\int_0^y \frac{1}{(x+y-2\xi)^{\frac{1}{2}}}\gamma(1,\frac{(x-\xi)(y-\xi)}{(x+y-2\xi)})d\xi & \text{if } y<x\end{cases}$$

Hence,

$$u(x,y)=\tfrac{1}{2}\begin{cases}\exp[-(y-x)]+\exp[-(x+y)]-\int_{(x+y)}^{(y-x)} t^{-\frac{1}{2}}\gamma(1,\frac{t^2-(x-y)^2}{4t})dt & \text{if } y>x\\ \exp[-(x-y)]+\exp[-(x+y)]-\int_{(x+y)}^{(x-y)} t^{-\frac{1}{2}}\gamma(1,\frac{t^2-(x-y)^2}{4t})dt & \text{if } y<x.\end{cases}$$

(b) Similarly, the transform equation for part (b) can be obtained

$$U(s_1,s_2)=\frac{1}{(s_1+1)(s_2+1)(s_1+s_2)}+\frac{4\pi^{\frac{1}{2}}}{3[1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]^{\frac{1}{2}}},$$

by virtue of Relation (2.2.i) in Example 2.2.2'.

Next, we combine Relations (1.2) in Ditkin and Prudnikov [43; p.100] and (2.2.ii) in Remark 2.2.2, we obtain the solution in the form

$$u(x,y)=\tfrac{1}{2}\begin{cases}\{\exp[-(y-x)]+\exp[-(x+y)]\}-\frac{3}{2}\int_{(x+y)}^{(y-x)} t^{-\frac{3}{2}}{}_2F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix} -\frac{t^2-(x-y)^2}{4t}\right]dt & \text{if } y>x\\ \{\exp[-(x-y)]+\exp[-(x+y)]\}-\frac{3}{2}\int_{(x+y)}^{(x-y)} t^{-\frac{3}{2}}{}_2F_1\left[\begin{matrix}\frac{3}{2}; \\ 1;\end{matrix} -\frac{t^2-(x-y)^2}{4t}\right]dt & \text{if } y<x.\end{cases}$$

Remark 4.3.2.3: It is easy to check that the condition of compatibility (4.3.2.3) holds true for the IBVPs in Example 4.3.2.3.

4.4. Non-homogenous Second order Partial Differential Equations of Parabolic Type

4.4.1. Partial Differential Equations of Type

$$u_{xx} + 2u_{xy} + u_{yy} + \kappa u = f(x,y),\ 0 < x < \infty,\ 0 < y < \infty,$$

where $\kappa = 0$ or 1.

Under the initial and boundary conditions

$$u(x,0) = u(0,y) = u_y(x,0) = u_x(0,y) = u(0,0) = 0 \tag{4.4.1}$$

Example 4.4.1.1. Determination of a solution $u = u(x,y)$ of (4.4.1) where $\kappa = 1$ and $u(x,0) = u(0,y) = u_y(x,0) = u_x(0,y) = u(0,0) = 0$, for

(a) $f(x,y) = (x+y)^{\frac{1}{2}}$

(b) $f(x,y) = \dfrac{(xy)^{\frac{\nu}{2}}}{(x+y)^{\frac{\nu+1}{2}}}$

(c) $f(x,y) = \dfrac{(xy)^{\frac{\nu}{2}}}{(x+y)^{\frac{\nu+3}{2}}}$

(d) $f(x,y) = \dfrac{1}{(x+y)^{\frac{1}{2}}}\exp[-\dfrac{xy}{x+y}]$

(a) Applying the double Laplace transform to (4.4.1a) with the aid of (1.5.2.8), (1.5.2.10), (1.5.2.11) and Relation (1.1‴) in Remark 2.1.1.2, we arrive at the following transform equation

$$U(s_1,s_2)=\frac{1}{(s_1+s_2)^2+1}\cdot\frac{\pi^{\frac{1}{2}}[s_1+s_2+(ss)^{\frac{1}{2}}]}{2(s_1s_2)^{\frac{3}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})},\quad \Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0. \tag{1.1a}$$

The inverse transform of (1.1a) can be obtained from (1.5.2.33) via Relation (1.1''') in Remark 2.1.1.2

$$u(x,y)=\int_0^x (x+y-2\xi)^{\frac{1}{2}}\sin\xi\, d\xi \tag{1.2a}$$

By a simple change of variable $x+y-2\xi=2t^2$ in (1.2), we obtain

$$u(x,y)=2^{-\frac{1}{2}}\int_{(\frac{x+y}{2})}^{(\frac{y-x}{2})} t^2\sin\left(t^2-\frac{x+y}{2}\right)dt \quad \text{if } y>x.$$

Expanding the sine and making some simplification, we arrive at

$$u(x,y)=2^{-\frac{1}{2}}[\cos\left(\frac{x+y}{2}\right)\int_{(\frac{x+y}{2})^{\frac{1}{2}}}^{(\frac{y-x}{2})^{\frac{1}{2}}} t^2\sin t^2 dt-\sin\left(\frac{x+y}{2}\right)\int_{(\frac{x+y}{2})^{\frac{1}{2}}}^{(\frac{y-x}{2})^{\frac{1}{2}}} t^2\cos t^2 dt] \tag{1.3a}$$

Using Formula 5 in Prudnikov et al. [84; p. 240], we obtain the solution in the form

$$u(x,y)=\frac{1}{4}\left\{\begin{array}{l}\left(\frac{x+y}{2}\right)^{\frac{1}{2}}-\left(\frac{y-x}{2}\right)^{\frac{1}{2}}\cos x+\pi^{\frac{1}{2}}\cos\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-C\left(\frac{x+y}{2}\right)\right]\\ \qquad +\pi^{\frac{1}{2}}\sin\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)-S\left(\frac{x+y}{2}\right)\right]\end{array}\right\} \text{ if } y>x.$$

Similarly, to obtain the transform equation for parts (b)-(d), respectively. We replace relation (1.1') in Example 1.1.1 which is used in part (a), Relations (1.1'') in Remark 2.1.1.2 for part (b), Formula 181 in Brychkov et al. [11; p. 300] for part (c) and (3.5'') in Example 3.2.3' for part (d), we arrive at.

$$U(s_1,s_2)=\frac{\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2}+1)}{[(s_1+s_2)^2+1](s_1s_2)^{\frac{1}{2}}(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\nu+1}} \tag{1.1b}$$

$$U(s_1,s_2)=\frac{\pi\Gamma(\nu+1)}{2^{\nu}\Gamma(\frac{\nu+3}{2})[(s_1+s_2)^2+1](s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})^{\nu+1}} \tag{1.1c}$$

$$U(s_1,s_2)=\frac{\pi(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})}{[(s_1+s_2)^2+1][1+(s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}})]^{\frac{1}{2}}}, \tag{1.1d}$$

where $\Re e[s_1^{\frac{1}{2}}+s_2^{\frac{1}{2}}]>0$.

Therefore, we obtain the following solutions, respectively

$$u(x,y)=\frac{1}{2^{\nu+1}}\int_{x+y}^{y-x} t^{-\frac{\nu+1}{2}}[t^2-(x-y)^2]^{\frac{\nu}{2}}\sin\left(\frac{t-(x+y)}{2}\right)dt \quad \text{if } y>x,\ \Re e\,\nu>-1$$

$$u(x,y)=\frac{1}{2^{\nu+1}}\int_{x+y}^{y-x} t^{-\frac{\nu+3}{2}}[t^2-(x-y)^2]^{\frac{\nu}{2}}\sin\left(\frac{t-(x+y)}{2}\right)dt \quad \text{if } y>x,\ \Re e\,\nu>-1$$

$$u(x,y)=\int_{x+y}^{y-x} t^{-\frac{1}{2}}\exp\left[\frac{t^2-(x+y)^2}{4t}\right]\sin\left(\frac{t-(x+y)}{2}\right)dt \quad \text{if } y>x.$$

Remark 4.4.1.1: Substituting $\nu=0$ in parts (b) and (c), leads to

$$u_{xx}+2u_{xy}+u_{yy}+u=\frac{1}{(x+y)^{\frac{1}{2}}}$$

$$u_{xx}+2u_{xy}+u_{yy}+u=\frac{1}{(x+y)^{\frac{3}{2}}}$$

Next, using (1.2b) and (1.2c), we arrive at the following solutions, respectively

$$u(x,y)=(\tfrac{\pi}{2})^{\frac{1}{2}}\left\{\cos\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)-S\left(\frac{x+y}{2}\right)\right]-\sin\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-C\left(\frac{x+y}{2}\right)\right]\right\} \quad \text{if } y>x.$$

$$\begin{aligned}u(x,y)=\frac{1}{(x+y)^{\frac{1}{2}}}\cos(x+y)-\frac{1}{(y-x)^{\frac{1}{2}}}\cos y-\pi^{\frac{1}{2}}\Big\{&\cos\left(\frac{x+y}{2}\right)\left[S\left(\frac{y-x}{2}\right)-S\left(\frac{x+y}{2}\right)\right]\\ &\sin\left(\frac{x+y}{2}\right)\left[C\left(\frac{y-x}{2}\right)-C\left(\frac{x+y}{2}\right)\right]\Big\} \quad \text{if } y>x.\end{aligned}$$

CHAPTER 5. CONCLUSIONS AND FUTURE DIRECTIONS

5.1. Conclusions

The main theme of this dissertation concerns the theoretical and computational aspects of N-dimensional Laplace transformation pairs, for $N \geq 2$. This was mainly done in Chapters 2 and 3. Laplace transforms can be defined either as a unilateral integral or a bilateral integral. We concentrated on the unilateral integrals. We have successfully developed a number of theorems and corollaries in N-dimensional Laplace transformations and inverse Laplace transformations. We have given numerous illustrative examples on applications of these results in N and two dimensions. We believe that these results will further enhance the use of N-dimensional Laplace transformation and help further development of more theoretical results.

Specifically, we established several two-dimensional Laplace transforms and inverse Laplace transforms in two-dimension pairs. These are in agreement with the results in Ditkin and Prudnikov [43], Voelker and Doetch [107] and Brychkov et al. [11], by taking the function to be the commonly used special functions.

Several initial boundary value problems (IBVPs) characterized by non-homogenous linear partial differential equations (PDEs) are explicitly solved in Chapter 4 by means of results established in Chapters 2 and 3. In the absence of necessitous three and N-dimensional Laplace transformation tables, we solved these IBVPs by the double Laplace transformations. These include non-homogenous linear PDEs of the first order, non-homogenous second order linear PDEs of Hyperbolic and Parabolic types.

5.2. Future Directions

Even though multi-dimensional Laplace transformations have been studied extensively since the early 1920s, or so, still a table of three or N-dimensional Laplace transforms is not available. To fill this gap much work is left to be done. To this end, we have developed several new results on N-dimensional Laplace transformations as well as inverse Laplace transformations and many more are still under our investigation. A successful completion of this task will be a significant endeavor, which will be extremely beneficial to the further research in Applied Mathematics, Engineering and Physical Sciences. Specifically, by the use of multi-dimensional Laplace transformations a PDE and its associated boundary conditions can be transformed into an algebraic equation in n independent variables, this algebraic equation can be solved to obtain the desired solution.

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